Question 1 (Jun 2006, Q9ii)

Worked Solution

Use the trapezium rule with 4 strips of width 0.5 to estimate the area bounded by $y = \left(\frac{1}{2}\right)^x$, the axes, and $x = 2$.

The x-values are 0, 0.5, 1, 1.5, 2 with $h = 0.5$.

Compute $y = \left(\frac{1}{2}\right)^x$ at each:

x	0	0.5	1	1.5	2
y	1	$0.5^{0.5}$	0.5	$0.5^{1.5}$	0.25

Note $0.5^{0.5} = \frac{1}{\sqrt{2}} \approx 0.70711$ and $0.5^{1.5} = \frac{1}{2\sqrt{2}} \approx 0.35355$.

Trapezium rule:

$$A \approx \frac{1}{2} \times 0.5 \times \left\{1 + 0.25 + 2(0.70711 + 0.5 + 0.35355)\right\}$$

$$= 0.25 \times \left\{1.25 + 2(1.56066)\right\} = 0.25 \times 4.37132 \approx 1.09$$

Area $\approx$ **1.09**

Question 2 (Jan 2007, Q5b)

Worked Solution

Use the trapezium rule with 2 strips of width 3 to estimate $\int_3^9 \log_{10} x \, dx$.

With $h = 3$, the three ordinates are at $x = 3, 6, 9$:

x	3	6	9
$y = \log_{10} x$	$\log_{10} 3$	$\log_{10} 6$	$\log_{10} 9$

Decimal values: $\log_{10} 3 \approx 0.47712$, $\log_{10} 6 \approx 0.77815$, $\log_{10} 9 \approx 0.95424$.

$$\int_3^9 \log_{10} x \, dx \approx \tfrac{1}{2} \times 3 \times \left\{ \log_{10} 3 + \log_{10} 9 + 2 \log_{10} 6 \right\}$$

$$= 1.5 \times \{0.47712 + 0.95424 + 2(0.77815)\} = 1.5 \times 2.98766 \approx 4.48$$

$\int_3^9 \log_{10} x \, dx \approx \mathbf{4.48}$ (3 s.f.)

Question 3 (Jan 2008, Q2)

Worked Solution

Use the trapezium rule with 3 strips each of width 2 to estimate $\int_1^7 \sqrt{x^2+3}\,dx$.

With $h = 2$, ordinates at $x = 1, 3, 5, 7$:

x	1	3	5	7
$y = \sqrt{x^2+3}$	$\sqrt{4} = 2$	$\sqrt{12}$	$\sqrt{28}$	$\sqrt{52}$

Decimal values: $\sqrt{12} \approx 3.464$, $\sqrt{28} \approx 5.292$, $\sqrt{52} \approx 7.211$.

$$\int_1^7 \sqrt{x^2+3}\,dx \approx \tfrac{1}{2} \times 2 \times \left\{2 + \sqrt{52} + 2\left(\sqrt{12} + \sqrt{28}\right)\right\}$$

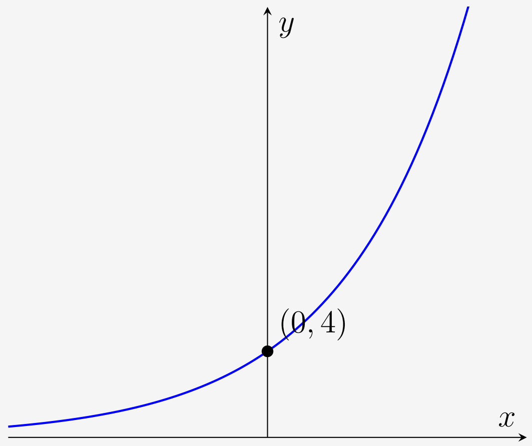
$$= 1 \times \{2 + 7.211 + 2(3.464 + 5.292)\} = 1 \times \{9.211 + 17.512\} \approx 26.7$$

$\int_1^7 \sqrt{x^2+3}\,dx \approx \mathbf{26.7}$ (3 s.f.)

Question 4 (Jun 2009, Q9)

Worked Solution

(i) Sketch: $y = 4k^x$ is an exponential growth curve ($k > 1$) passing through $(0, 4)$. It crosses the y-axis at $(0, 4)$ only; it does not cross the x-axis.//



(ii) $4k^x = 20k^2 \Rightarrow k^x = 5k^2 \Rightarrow k^{x-2} = 5$.

Taking $\log_k$: $x - 2 = \log_k 5$, so $x = 2 + \log_k 5$. □

(iii)(a) With $h = \frac{1}{2}$, two strips from $x = 0$ to $x = 1$. Ordinates at $x = 0$, $\frac{1}{2}$, 1:

$$y(0) = 4k^0 = 4, \quad y\left(\tfrac{1}{2}\right) = 4k^{1/2} = 4\sqrt{k}, \quad y(1) = 4k$$

$$\int_0^1 4k^x \, dx \approx \tfrac{1}{2} \times \tfrac{1}{2} \times \left\{4 + 4k + 2(4\sqrt{k})\right\} = \tfrac{1}{4}\left(4 + 4k + 8\sqrt{k}\right)$$

$$\int_0^1 4k^x \, dx \approx 1 + k + 2\sqrt{k}$$

(iii)(b) Set the approximation equal to 16:

$$1 + k + 2\sqrt{k} = 16 \implies k + 2\sqrt{k} - 15 = 0$$

Let $t = \sqrt{k}$: $t^2 + 2t - 15 = 0 \implies (t + 5)(t - 3) = 0$.

Since $k > 1$, $t = \sqrt{k} > 1$, so $t = 3$ and $k = 9$.

$$k = 9$$

Question 5 (Jan 2010, Q4)

Worked Solution

(i) With $h = 0.5$, 4 strips from $x = 3$ to $x = 5$. Ordinates at $x = 3, 3.5, 4, 4.5, 5$:

x	3	3.5	4	4.5	5
$y = \log_{10}(2+x)$	$\log_{10} 5$	$\log_{10} 5.5$	$\log_{10} 6$	$\log_{10} 6.5$	$\log_{10} 7$

Decimal values: $0.69897,\ 0.74036,\ 0.77815,\ 0.81291,\ 0.84510.$

$$\int_3^5 \log_{10}(2+x)\,dx \approx \tfrac{1}{2} \times 0.5 \times \Big\{0.69897 + 0.84510 + 2(0.74036 + 0.77815 + 0.81291)\Big\}$$

$$= 0.25 \times \{1.54407 + 2(2.33142)\} = 0.25 \times 6.20691 \approx 1.55$$

$$\int_3^5 \log_{10}(2+x)\,dx \approx \mathbf{1.55} \text{ (3 s.f.)}$$

(ii) Note $\log_{10}\sqrt{2+x} = \tfrac{1}{2}\log_{10}(2+x)$, so:

$$\int_3^5 \log_{10}\sqrt{2+x}\,dx = \frac{1}{2}\int_3^5 \log_{10}(2+x)\,dx \approx \frac{1}{2} \times 1.55 = 0.775$$

$$\int_3^5 \log_{10}\sqrt{2+x}\,dx \approx \mathbf{0.78} \text{ (2 s.f.)}$$

Question 6 (Jun 2013, Q1)

Worked Solution

Use the trapezium rule with 3 strips each of width 2 to estimate $\int_5^{11} \frac{8}{x}\,dx$.

With $h = 2$, ordinates at $x = 5, 7, 9, 11$:

x	5	7	9	11
$y = \frac{8}{x}$	$\frac{8}{5}$	$\frac{8}{7}$	$\frac{8}{9}$	$\frac{8}{11}$

$$\int_5^{11} \frac{8}{x}\,dx \approx \tfrac{1}{2} \times 2 \times \left\{\frac{8}{5} + \frac{8}{11} + 2\left(\frac{8}{7} + \frac{8}{9}\right)\right\}$$

$$= 1 \times \{1.6 + 0.7273 + 2(1.1429 + 0.8889)\} = 1 \times \{2.3273 + 4.0635\} \approx 6.39$$

$\int_5^{11} \frac{8}{x}\,dx \approx \mathbf{6.39}$ (3 s.f.)

Question 7 (Jun 2014, Q9i,ii)

Worked Solution

Curve: $y = -3 + 2\sqrt{x+4}$. Region A bounded by curve, x-axis, y-axis, $x = 5$.

(i) With $h = 2.5$, 2 strips from $x = 0$ to $x = 5$. Ordinates at $x = 0, 2.5, 5$:

$$y(0) = -3 + 2\sqrt{4} = -3 + 4 = 1$$

$$y(2.5) = -3 + 2\sqrt{6.5} = -3 + 2(2.5495\ldots) = -3 + 5.0990 = 2.0990\ldots \approx 2.099$$

$$y(5) = -3 + 2\sqrt{9} = -3 + 6 = 3$$

$$\text{Area } A \approx \tfrac{1}{2} \times 2.5 \times \{1 + 3 + 2(2.099)\} = 1.25 \times \{4 + 4.198\} = 1.25 \times 8.198 \approx 10.2$$

Area of $A \approx$ **10.2** (3 s.f.)

(ii) The rectangle from $x = 0$ to $x = 5$, $y = 0$ to $y = 3$ has area $5 \times 3 = 15$.

Region B is bounded by the curve, the y-axis and $y = 3$:

$$\text{Area } B = \text{Rectangle} - \text{Area } A \approx 15 - 10.2 = 4.8$$

Area of $B \approx$ **4.8** (2 s.f.)

Question 8 (Jun 2015, Q2)

Worked Solution

(i) With $h = 1.5$, 4 strips from $x = 4$ to $x = 10$. Ordinates at $x = 4, 5.5, 7, 8.5, 10$:

x	4	5.5	7	8.5	10
$y = \sqrt{2x-1}$	$\sqrt{7}$	$\sqrt{10}$	$\sqrt{13}$	$\sqrt{16}$	$\sqrt{19}$

Decimal values: 2.6458, 3.1623, 3.6056, 4, 4.3589.

$$\int_4^{10} \sqrt{2x-1}\,dx \approx \tfrac{1}{2} \times 1.5 \times \left\{\sqrt{7} + \sqrt{19} + 2(\sqrt{10} + \sqrt{13} + \sqrt{16})\right\}$$
$$= 0.75 \times \{2.6458 + 4.3589 + 2(3.1623 + 3.6056 + 4)\}$$
$$= 0.75 \times \{7.0047 + 2(10.7679)\} = 0.75 \times 28.5405 \approx 21.4$$

$$\int_4^{10} \sqrt{2x-1}\,dx \approx \mathbf{21.4} \text{ (3 s.f.)}$$

(ii) To obtain a more accurate estimate, use **more strips** (smaller strip width).

Question 9 (Jun 2016, Q8v)

Worked Solution

Use the trapezium rule with 2 strips each of width 1.5 to estimate $\int_1^4 3^{x-2}\,dx$.

With $h = 1.5$, ordinates at $x = 1,\ 2.5,\ 4$:

$$y(1) = 3^{-1} = \tfrac{1}{3}, \quad y(2.5) = 3^{0.5} = \sqrt{3}, \quad y(4) = 3^2 = 9$$

Wait — note $3^{0.5} = \sqrt{3} \approx 1.7321$ and $3^2 = 9$ would give very large strips. Let us recalculate carefully:

$x = 1$: $y = 3^{1-2} = 3^{-1} = 0.3333$

$x = 2.5$: $y = 3^{2.5-2} = 3^{0.5} = \sqrt{3} \approx 1.7321$

$x = 4$: $y = 3^{4-2} = 3^2 = 9$

$$\int_1^4 3^{x-2}\,dx \approx \tfrac{1}{2} \times 1.5 \times \left\{3^{-1} + 3^2 + 2 \cdot 3^{0.5}\right\}$$

$$= 0.75 \times \left\{\frac{1}{3} + 9 + 2\sqrt{3}\right\} = 0.75 \times \{0.3333 + 9 + 3.4641\} = 0.75 \times 12.7974 \approx 9.60$$

$\int_1^4 3^{x-2}\,dx \approx \mathbf{9.60}$ (3 s.f.)

End of Worked Solutions