

Question 1

Worked Solution

(a) We need y when $x = 4$ and $x = 8$ for $y = \sqrt{2x - 1}$.

When $x = 4$: $y = \sqrt{2(4) - 1} = \sqrt{7}$

When $x = 8$: $y = \sqrt{2(8) - 1} = \sqrt{15}$

$$y(4) = \sqrt{7} \approx 2.646 \quad \text{and} \quad y(8) = \sqrt{15} \approx 3.873$$

(b) The completed table is:

| | | | | | |
|-----|------------|------------|-------------|-------------|-------------|
| x | 2 | 4 | 6 | 8 | 10 |
| y | $\sqrt{3}$ | $\sqrt{7}$ | $\sqrt{11}$ | $\sqrt{15}$ | $\sqrt{19}$ |

With $n = 4$ strips and $h = 2$, the trapezium rule gives:

$$\begin{aligned} \text{Area} &\approx \frac{1}{2} \times 2 \times \left\{ \sqrt{3} + \sqrt{19} + 2(\sqrt{7} + \sqrt{11} + \sqrt{15}) \right\} \\ &= 1 \times \left\{ 1.7321 + 4.3589 + 2(2.6458 + 3.3166 + 3.8730) \right\} \\ &= 1 \times \left\{ 6.0909 + 2(9.8354) \right\} = 1 \times 25.7617 \dots \end{aligned}$$

$$\text{Area} \approx \mathbf{25.76} \text{ (2 d.p.)}$$

(c) The curve $y = \sqrt{2x - 1}$ is concave (curves downwards) on this interval, meaning the trapezium chords lie *below* the curve. Therefore the trapezium rule gives an **underestimate**.

Question 2

Worked Solution

(a) The table is:

| | | | | | |
|-----|------|-----|------|------|------|
| x | 3 | 4.5 | 6 | 7.5 | 9 |
| y | 1.63 | 2 | 2.26 | 2.46 | 2.63 |

With $n = 4$ strips and $h = 1.5$:

$$\int_3^9 \log_3 2x \, dx \approx \frac{1.5}{2} \{1.63 + 2.63 + 2(2 + 2.26 + 2.46)\}$$

$$= 0.75 \times \{4.26 + 2(6.72)\} = 0.75 \times 17.70 = 13.275$$

$$\int_3^9 \log_3 2x \, dx \approx 13.3 \text{ (3 s.f.)} \quad (\text{exact: } \frac{531}{40})$$

(b)(i) Using $\log_3(2x)^{10} = 10 \log_3 2x$:

$$\int_3^9 \log_3(2x)^{10} \, dx = 10 \int_3^9 \log_3 2x \, dx \approx 10 \times 13.275$$

$$\int_3^9 \log_3(2x)^{10} \, dx \approx 133 \text{ (3 s.f.)}$$

(b)(ii) Write $\log_3 18x = \log_3(9 \times 2x) = \log_3 9 + \log_3 2x = 2 + \log_3 2x$.

$$\int_3^9 \log_3 18x \, dx = \int_3^9 2 \, dx + \int_3^9 \log_3 2x \, dx = [2x]_3^9 + 13.275$$

$$= (18 - 6) + 13.275 = 12 + 13.275 = 25.275$$

$$\int_3^9 \log_3 18x \, dx \approx 25.3 \text{ (3 s.f.)}$$

Question 3

Worked Solution

(a) Values of $y = \sqrt{2^x + 1}$ to 3 d.p.:

When $x = 2$: $y = \sqrt{4 + 1} = \sqrt{5} = 2.236$

When $x = 2.5$: $y = \sqrt{2^{2.5} + 1} = \sqrt{5.6569\dots + 1} = \sqrt{6.6569\dots} = 2.580$

$$y(2) = 2.236 \quad \text{and} \quad y(2.5) = 2.580$$

(b) Completed table:

| | | | | | | | |
|-----|-------|-------|-------|-------|-------|-------|---|
| x | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
| y | 1.414 | 1.554 | 1.732 | 1.957 | 2.236 | 2.580 | 3 |

With $n = 6$ strips and $h = 0.5$:

$$\begin{aligned} \text{Area} &\approx \frac{1}{2} \times \frac{1}{2} \times \left\{ (1.414 + 3) + 2(1.554 + 1.732 + 1.957 + 2.236 + 2.580) \right\} \\ &= \frac{1}{4} \times \left\{ 4.414 + 2(10.059) \right\} = \frac{1}{4} \times 24.532 = 6.133 \end{aligned}$$

$$\text{Area} \approx \mathbf{6.133}$$

(c) The curve $y = \sqrt{2^x + 1}$ is **convex** (curves upwards) on $[0, 3]$, so the trapezia lie **above** the curve. Therefore the approximation is an **overestimate**.

Question 4

Worked Solution

(a) When $x = 1$: $y = 5^1 + \log_2(1 + 1) = 5 + \log_2 2 = 5 + 1 = 6$

$$y(1) = 6$$

(b) Completed table:

| | | | | | |
|-----|---|-------|---|--------|--------|
| x | 0 | 0.5 | 1 | 1.5 | 2 |
| y | 1 | 2.821 | 6 | 12.502 | 26.585 |

With $n = 4$ strips and $h = 0.5$:

$$\begin{aligned} \int_0^2 (5^x + \log_2(x + 1)) dx &\approx \frac{1}{2} \times 0.5 \times \{1 + 26.585 + 2(2.821 + 6 + 12.502)\} \\ &= 0.25 \times \{27.585 + 2(21.323)\} = 0.25 \times 70.231 = 17.5578 \dots \end{aligned}$$

$$\int_0^2 (5^x + \log_2(x + 1)) dx \approx \mathbf{17.56} \text{ (2 d.p.)}$$

(c) Adding 5 to the integrand adds $5 \times (2 - 0) = 10$ to the integral:

$$\int_0^2 (5 + 5^x + \log_2(x + 1)) dx \approx 10 + 17.56$$

$$\int_0^2 (5 + 5^x + \log_2(x + 1)) dx \approx \mathbf{27.56} \text{ (2 d.p.)}$$

Question 5

Worked Solution

(a) With $h = 0.2$, $n = 5$ strips from $x = 3$ to $x = 4$:

$$\text{Area} \approx \frac{1}{2} \times 0.2 \times \{a + 13.5 + 2(16.8 + b + 20.2 + 18.7)\} = 17.59$$

$$0.1 \times \{a + 13.5 + 2(55.7 + b)\} = 17.59$$

$$a + 13.5 + 111.4 + 2b = 175.9$$

$$\therefore a + 2b + 124.9 = 175.9$$

$$a + 2b = 51 \quad (\text{as required})$$

(b) Also, the sum of all y -values is 97.2:

$$a + 16.8 + b + 20.2 + 18.7 + 13.5 = 97.2$$

$$a + b + 69.2 = 97.2$$

$$a + b = 28 \quad \dots (1)$$

From part (a): $a + 2b = 51 \quad \dots (2)$

Subtracting (1) from (2): $b = 23$, then $a = 28 - 23 = 5$.

$$a = 5 \quad \text{and} \quad b = 23$$

Question 6

Worked Solution

(a) With $h = 0.5$, $n = 4$ strips:

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}} dx \approx \frac{0.5}{2} \{0.5774 + 0.8452 + 2(0.7071 + 0.7746 + 0.8165)\}$$

$$= 0.25 \times \{1.4226 + 2(2.2982)\} = 0.25 \times 5.9226 \dots \approx 1.4807 \dots$$

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}} dx \approx \mathbf{1.50} \text{ (3 s.f.)}$$

(b) Note that $\sqrt{\frac{9x}{1+x}} = 3\sqrt{\frac{x}{1+x}}$, so:

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}} dx = 3 \int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}} dx \approx 3 \times 1.50 = 4.50$$

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}} dx \approx \mathbf{4.50} \text{ (3 s.f.)}$$

(c) The given exact value is 4.535 (4 s.f.). Our estimate of 4.50 differs by $4.535 - 4.50 = 0.035$, a percentage error of $\frac{0.035}{4.535} \times 100 \approx 0.77\%$. The estimate is accurate to 2 significant figures / within 1% of the true value, so it is a **good approximation**.

Question 7

Worked Solution

(a) With $h = 0.5$, $n = 4$ strips:

$$\begin{aligned} \text{Area of } S &\approx \frac{0.5}{2} \left\{ 3 + 2.2958 + 2(2.3041 + 1.9242 + 1.9089) \right\} \\ &= 0.25 \times \{ 5.2958 + 2(6.1372) \} = 0.25 \times 17.5702 = 4.3926 \dots \end{aligned}$$

Area of $S \approx \mathbf{4.393}$ (3 d.p.)

(b) To obtain a more accurate estimate, **increase the number of strips** (use a smaller value of h).

(c) Integrate $y = \frac{x^2 \ln x}{3} - 2x + 5$ from $x = 1$ to $x = 3$.

First integrate $\int x^2 \ln x \, dx$ by parts with $u = \ln x$, $dv = x^2 \, dx$:

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9}$$

Also $\int (-2x + 5) \, dx = -x^2 + 5x$.

Therefore:

$$\text{Area of } S = \int_1^3 \left(\frac{x^2 \ln x}{3} - 2x + 5 \right) dx = \left[\frac{x^3}{9} \ln x - \frac{x^3}{27} - x^2 + 5x \right]_1^3$$

At $x = 3$: $\frac{27}{9} \ln 3 - \frac{27}{27} - 9 + 15 = 3 \ln 3 - 1 - 9 + 15 = 3 \ln 3 + 5$

At $x = 1$: $\frac{1}{9} \ln 1 - \frac{1}{27} - 1 + 5 = 0 - \frac{1}{27} + 4 = 4 - \frac{1}{27} = \frac{107}{27}$

$$\text{Area} = (3 \ln 3 + 5) - \frac{107}{27} = 3 \ln 3 + \frac{135 - 107}{27} = 3 \ln 3 + \frac{28}{27}$$

Using $\ln 27 = 3 \ln 3$:

Area of $S = \frac{28}{27} + \ln 27$ ($a = 28$, $b = 27$, $c = 27$)

Question 8

Worked Solution

(a) When $t = 6$: $x = 4(6)e^{-6/3} + 3 = 24e^{-2} + 3 = 24 \times 0.13534 \dots + 3 = 3.2481 \dots + 3 = 6.248$ (3 d.p.)

$$x(6) = 6.248$$

(b) With $h = 2$, $n = 4$ strips:

$$\begin{aligned} \text{Area} &\approx \frac{1}{2} \times 2 \times \left\{ 3 + 5.223 + 2(7.107 + 7.218 + 6.248) \right\} \\ &= 1 \times \{ 8.223 + 2(20.573) \} = 1 \times 49.369 = 49.37 \text{ (2 d.p.)} \end{aligned}$$

$$\text{Area} \approx \mathbf{49.37} \text{ (2 d.p.)}$$

(c) Integrate $x = 4te^{-t/3} + 3$ from $t = 0$ to $t = 8$ by parts.

Let $u = 4t$, $dv = e^{-t/3}dt$, so $du = 4dt$, $v = -3e^{-t/3}$:

$$\int 4te^{-t/3} dt = -12te^{-t/3} + \int 12e^{-t/3} dt = -12te^{-t/3} - 36e^{-t/3}$$

$$\int_0^8 (4te^{-t/3} + 3) dt = [-12te^{-t/3} - 36e^{-t/3} + 3t]_0^8$$

At $t = 8$: $-12(8)e^{-8/3} - 36e^{-8/3} + 24 = -96e^{-8/3} - 36e^{-8/3} + 24 = -132e^{-8/3} + 24$

At $t = 0$: $0 - 36(1) + 0 = -36$

$$\text{Area} = (-132e^{-8/3} + 24) - (-36) = 60 - 132e^{-8/3}$$

$$\text{Exact area} = 60 - 132e^{-8/3}$$

(d) $60 - 132e^{-8/3} = 60 - 132(0.06948 \dots) = 60 - 9.172 \dots = 50.828 \dots$

Difference = $|50.83 - 49.37| = 1.46$ (2 d.p.)

$$\text{Difference} = \mathbf{1.46} \text{ (2 d.p.)}$$

Question 9

Worked Solution

(a) When $x = 3$: $y = \frac{10}{2(3) + 5\sqrt{3}} = \frac{10}{6 + 5(1.73205\dots)} = \frac{10}{6 + 8.66025\dots} = \frac{10}{14.66025\dots} = 0.68212$ (5 d.p.)

$$y(3) = 0.68212$$

(b) Completed table:

| | | | | |
|-----|---------|---------|---------|---------|
| x | 1 | 2 | 3 | 4 |
| y | 1.42857 | 0.90326 | 0.68212 | 0.55556 |

With $h = 1$, $n = 3$ strips:

$$\begin{aligned} \int_1^4 \frac{10}{2x + 5\sqrt{x}} dx &\approx \frac{1}{2} \times 1 \times \{1.42857 + 0.55556 + 2(0.90326 + 0.68212)\} \\ &= 0.5 \times \{1.98413 + 2(1.58538)\} = 0.5 \times 5.15489 = 2.5774 \end{aligned}$$

Area \approx **2.5774** (4 d.p.)

(c) The curve $y = \frac{10}{2x + 5\sqrt{x}}$ is convex (concave upward, $y'' > 0$) on $[1, 4]$, so the trapezia lie **above** the curve. The estimate is an **overestimate**.

(d) Let $u = \sqrt{x}$, so $x = u^2$, $dx = 2u du$. When $x = 1$, $u = 1$; when $x = 4$, $u = 2$.

$$\begin{aligned} \int_1^4 \frac{10}{2x + 5\sqrt{x}} dx &= \int_1^2 \frac{10}{2u^2 + 5u} \cdot 2u du = \int_1^2 \frac{20u}{u(2u + 5)} du = \int_1^2 \frac{20}{2u + 5} du \\ &= [10 \ln(2u + 5)]_1^2 = 10 \ln 9 - 10 \ln 7 \end{aligned}$$

Exact value = $10 \ln 9 - 10 \ln 7 = 10 \ln\left(\frac{9}{7}\right)$

Question 10

Worked Solution

(a) When $x = 0.2$: $y = \frac{6}{e^{0.2} + 2} = \frac{6}{1.22140\dots + 2} = \frac{6}{3.22140\dots} = 1.86254$ (5 d.p.)

$$y(0.2) = 1.86254$$

(b) With $h = 0.2$, $n = 5$ strips:

$$\begin{aligned} \int_0^1 \frac{6}{e^x + 2} dx &\approx \frac{1}{2} \times 0.2 \times \left\{ 2 + 1.27165 + 2(1.86254 + 1.71830 + 1.56981 + 1.41994) \right\} \\ &= 0.1 \times \{ 3.27165 + 2(6.57059) \} = 0.1 \times 16.41283 = 1.6413 \end{aligned}$$

$$\text{Area} \approx \mathbf{1.6413} \text{ (4 d.p.)}$$

(c) Let $u = e^x$, so $\frac{du}{dx} = e^x = u$, i.e. $dx = \frac{du}{u}$.

When $x = 0$: $u = 1$; when $x = 1$: $u = e$.

$$\int_0^1 \frac{6}{e^x + 2} dx = \int_1^e \frac{6}{u + 2} \cdot \frac{du}{u} = \int_1^e \frac{6}{u(u + 2)} du$$

$$\int_1^e \frac{6}{u(u + 2)} du \quad \text{with } a = 1, b = e$$

(d) Partial fractions: $\frac{6}{u(u + 2)} = \frac{3}{u} - \frac{3}{u + 2}$

$$\begin{aligned} \int_1^e \frac{6}{u(u + 2)} du &= \left[3 \ln u - 3 \ln(u + 2) \right]_1^e \\ &= (3 \ln e - 3 \ln(e + 2)) - (3 \ln 1 - 3 \ln 3) \\ &= 3 - 3 \ln(e + 2) - 0 + 3 \ln 3 \end{aligned}$$

$$\text{Exact area} = 3 + 3 \ln 3 - 3 \ln(e + 2) = 3 + 3 \ln \left(\frac{3}{e + 2} \right)$$

End of Worked Solutions