Question 1 (Jan 2006, Q2)

Worked Solution

Use the Newton–Raphson method to find the root of

$$e^{-x} = x$$

which is close to $x = 0.5$. Give the root correct to 3 decimal places.

Write the equation in the form $f(x) = 0$. A convenient choice is

$$f(x) = x - e^{-x}.$$

Then

$$f'(x) = 1 + e^{-x}.$$

So the Newton–Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}.$$

The root is stated to be close to 0.5, so take

$$x_0 = 0.5.$$

Then

$$x_1 = 0.5 - \frac{0.5 - e^{-0.5}}{1 + e^{-0.5}} = 0.5 - \frac{0.5 - 0.6065307}{1 + 0.6065307} \approx 0.5663110.$$

Next,

$$x_2 = 0.5663110 - \frac{0.5663110 - e^{-0.5663110}}{1 + e^{-0.5663110}} \approx 0.5671432.$$

Next,

$$x_3 = 0.5671432 - \frac{0.5671432 - e^{-0.5671432}}{1 + e^{-0.5671432}} \approx 0.5671433.$$

Since successive values agree to 3 decimal places, the required root is

$$x \approx 0.567$$

(to 3 d.p.)

Question 2 (Jan 2008, Q5)

Worked Solution

The curve has equation

$$y = xe^{-x} + 1,$$

and crosses the x-axis at $x = \alpha$.

(i) Show that the x-coordinate of the stationary point is 1.

Let

$$f(x) = xe^{-x} + 1.$$

Differentiate using the product rule:

$$f'(x) = x(-e^{-x}) + e^{-x} = e^{-x}(1 - x).$$

At a stationary point, $f'(x) = 0$, so

$$e^{-x}(1 - x) = 0.$$

Since $e^{-x} \neq 0$ for all real x, we must have

$$1 - x = 0 \quad \Rightarrow \quad x = 1.$$

The stationary point has x-coordinate 1.

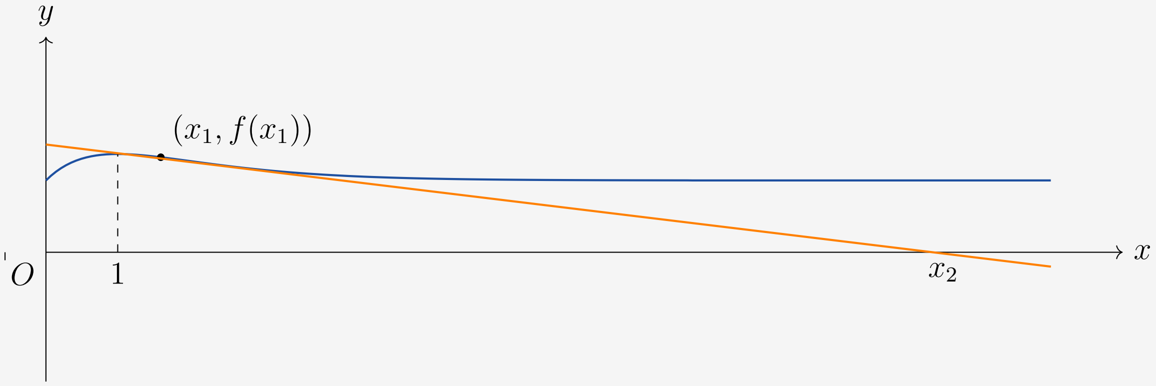
(ii) Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$.

From part (i), the curve has a stationary point at $x = 1$, so the tangent there is horizontal. For $x > 1$ we have

$$f'(x) = e^{-x}(1 - x) < 0,$$

so the tangents have negative gradient, and near $x = 1$ they are quite shallow.

The root α lies to the *left* of 1. If we start with a value $x_1 > 1$, the tangent at $(x_1, f(x_1))$ meets the x-axis at a point still to the right of the root, giving another approximation with $x > 1$. Repeating this means the successive tangents keep producing x-intercepts on the wrong side of the root, so the iteration does not converge to α.



If $x_1 > 1$, the successive tangents meet the x-axis at points on the wrong side of the root, so the Newton–Raphson method does not converge to α.

(iii) Using $x_1 = 0$, find the next three approximations x_2, x_3 and x_4. Hence find α, correct to 3 decimal places.

We use

$$f(x) = xe^{-x} + 1, \qquad f'(x) = e^{-x}(1 - x).$$

So

$$x_{n+1} = x_n - \frac{xe^{-x} + 1}{e^{-x}(1 - x)}.$$

Starting with $x_1 = 0$,

$$f(0) = 1, \qquad f'(0) = 1,$$

so

$$x_2 = 0 - \frac{1}{1} = -1.$$

Now use $x_2 = -1$:

$$f(-1) = -e + 1, \qquad f'(-1) = 2e.$$

Hence

$$x_3 = -1 - \frac{-e + 1}{2e} = -1 + \frac{e - 1}{2e} \approx -0.6839397.$$

Now use $x_3 \approx -0.6839397$:

$$x_4 = x_3 - \frac{x_3 e^{-x_3} + 1}{e^{-x_3}(1 - x_3)} \approx -0.5774545.$$

Continuing once more,

$$x_5 \approx -0.5672297,$$

so the root is

$$\alpha \approx -0.567$$

(to 3 d.p.)

Question 3 (Jun 2008, Q6i,ii)

Worked Solution

It is given that

$$f(x) = 1 - \frac{7}{x^2}.$$

(i) Use the Newton–Raphson method, with first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 6 decimal places.

Differentiate:

$$f'(x) = 14x^{-3} = \frac{14}{x^3}.$$

Hence

$$x_{n+1} = x_n - \frac{1 - 7/x_n^2}{14/x_n^3}.$$

Using $x_1 = 2.5$,

$$f(2.5) = 1 - \frac{7}{(2.5)^2} = 1 - \frac{7}{6.25} = -0.12,$$

$$f'(2.5) = \frac{14}{(2.5)^3} = \frac{14}{15.625} = 0.896.$$

Therefore

$$x_2 = 2.5 - \frac{-0.12}{0.896} = 2.633928571\ldots$$

so

$$x_2 = 2.633929.$$

Now use $x_2 = 2.633928571\ldots$:

$$x_3 = x_2 - \frac{1 - 7/x_2^2}{14/x_2^3} \approx 2.645672182\ldots$$

so

$$x_3 = 2.645672.$$

$$x_2 = 2.633929, \qquad x_3 = 2.645672$$

(to 6 d.p.)

(ii) The root of $f(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α.

Set $f(x) = 0$:

$$1 - \frac{7}{x^2} = 0 \quad \Rightarrow \quad \frac{7}{x^2} = 1 \quad \Rightarrow \quad x^2 = 7.$$

So $x = \pm\sqrt{7}$. Since the iteration starts from the positive value $x_1 = 2.5$, it is converging to the positive root.

$$\boxed{\alpha = \sqrt{7}}$$

Question 4 (Jan 2010, Q3i,ii)

Worked Solution

A curve with no stationary points has equation $y = f(x)$. The equation $f(x) = 0$ has one real root α, and the tangent to the curve at the point $(x_1, f(x_1))$ meets the x-axis where $x = x_2$.

(i) Show that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The tangent at $(x_1, f(x_1))$ has gradient $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

At the point where this tangent meets the x-axis, $y = 0$ and $x = x_2$. Therefore

$$0 - f(x_1) = f'(x_1)(x_2 - x_1).$$

So

$$-f(x_1) = f'(x_1)(x_2 - x_1),$$

which gives

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}.$$

Hence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The Newton–Raphson formula follows directly from the equation of the tangent.

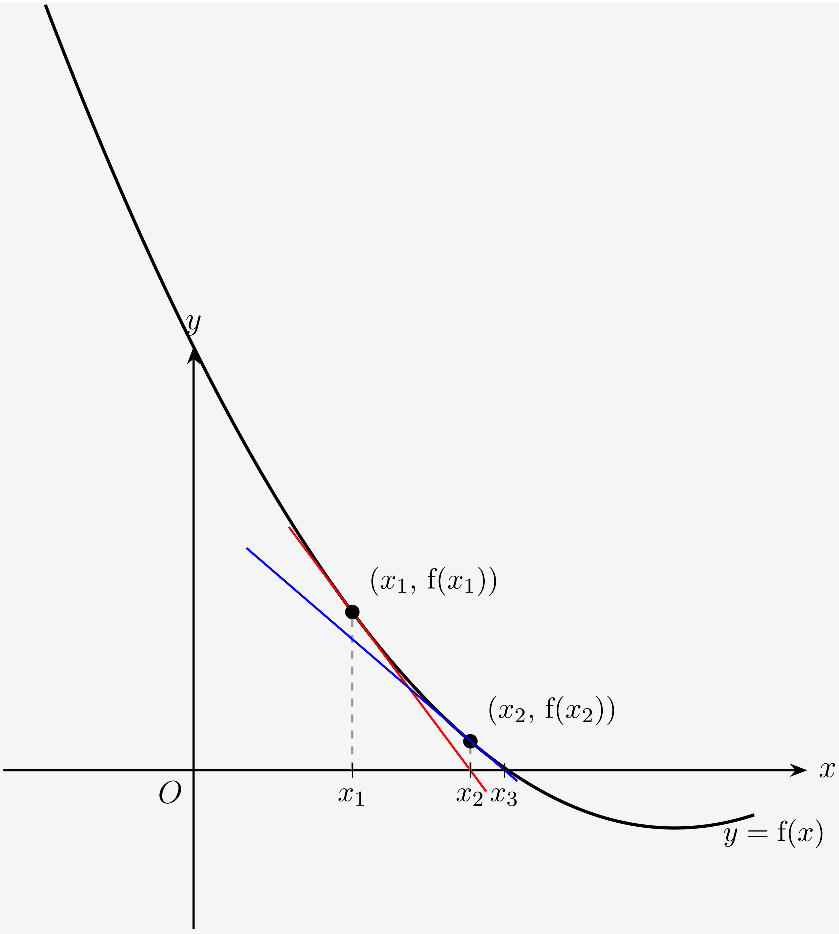
(ii) Describe briefly, with the help of a sketch, how the Newton–Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching α.

Starting at $x = x_1$, move vertically to the curve to the point $(x_1, f(x_1))$. Draw the tangent at this point. Where this tangent meets the x-axis gives the next approximation, x_2.

Then repeat the process: move vertically from x_2 to the curve, draw the tangent there, and the point where that tangent meets the x-axis is x_3. Continuing in this way gives a sequence

$$x_1,\ x_2,\ x_3,\ \ldots$$

which approaches the root α.



Each tangent gives the next x-intercept, which is used as the next approximation. Repeating this process generates values that approach the root α.

Question 5 (Jun 2010, Q7iii,iv)

Worked Solution

The line $y = x$ and the curve $y = 2\ln(3x-2)$ meet where $x = \alpha$ and $x = \beta$.

(iii) Show that the equation $x = 2\ln(3x-2)$ can be rewritten as

$$x = \frac{1}{3}\left(e^{x/2} + 2\right).$$

Use the Newton–Raphson method, with

$$f(x) = \frac{1}{3}\left(e^{x/2} + 2\right) - x$$

and $x_1 = 1.2$, to find α correct to 2 decimal places.

Starting with

$$x = 2\ln(3x-2),$$

divide by 2:

$$\frac{x}{2} = \ln(3x-2).$$

Now exponentiate:

$$e^{x/2} = 3x - 2.$$

So

$$3x = e^{x/2} + 2,$$

and therefore

$$x = \frac{1}{3}\left(e^{x/2} + 2\right).$$

This is the required rearrangement.

Now define

$$f(x) = \frac{1}{3}\left(e^{x/2} + 2\right) - x.$$

Then

$$f'(x) = \frac{1}{6}e^{x/2} - 1.$$

So Newton–Raphson gives

$$x_{n+1} = x_n - \frac{\frac{1}{3}(e^{x_n/2} + 2) - x_n}{\frac{1}{6}e^{x_n/2} - 1}.$$

Using $x_1 = 1.2$,

$$f(1.2) = \frac{1}{3}(e^{0.6} + 2) - 1.2 \approx \frac{1}{3}(1.8221188 + 2) - 1.2 \approx 0.0740396,$$

$$f'(1.2) = \frac{1}{6}e^{0.6} - 1 \approx \frac{1}{6}(1.8221188) - 1 \approx -0.6963135.$$

Hence

$$x_2 = 1.2 - \frac{0.0740396}{-0.6963135} \approx 1.3063308.$$

Next,

$$x_3 \approx 1.3076164,$$

and then

$$x_4 \approx 1.3076165.$$

So the root is

$$\boxed{\alpha \approx 1.31}$$

(to 2 d.p.)

(iv) Given that $x_1 = \ln 36$, explain why the Newton–Raphson method would not converge to a root of $f(x) = 0$.

We have

$$f'(x) = \frac{1}{6}e^{x/2} - 1.$$

Now if $x = \ln 36$, then

$$e^{x/2} = e^{(\ln 36)/2} = \sqrt{36} = 6.$$

Therefore

$$f'(\ln 36) = \frac{1}{6} \cdot 6 - 1 = 0.$$

So at $x_1 = \ln 36$ the tangent is horizontal. In the Newton–Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

this would mean dividing by 0, so the method breaks down. Equivalently, a horizontal tangent does not meet the x-axis again.

At $x_1 = \ln 36$, $f'(x_1) = 0$, so the tangent is parallel to the x-axis and the Newton–Raphson formula is undefined.

Question 6 (Jan 2011, Q5)

Worked Solution

The equation

$$x^3 - 5x + 3 = 0 \tag{A}$$

may be solved by the Newton–Raphson method.

(i) Show that the Newton–Raphson formula can be written in the form $x_{n+1} = F(x_n)$, where

$$F(x) = \frac{2x^3 - 3}{3x^2 - 5}.$$

Take

$$f(x) = x^3 - 5x + 3,$$

so

$$f'(x) = 3x^2 - 5.$$

Newton–Raphson gives

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5}.$$

Write this as a single fraction:

$$x_{n+1} = \frac{x_n(3x_n^2 - 5) - (x_n^3 - 5x_n + 3)}{3x_n^2 - 5}.$$

Simplifying the numerator,

$$3x_n^3 - 5x_n - x_n^3 + 5x_n - 3 = 2x_n^3 - 3.$$

Hence

$$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 5}.$$

So indeed

$$F(x) = \frac{2x^3 - 3}{3x^2 - 5}.$$

$$\boxed{F(x) = \frac{2x^3 - 3}{3x^2 - 5}}$$

(ii) Find $F'(x)$ and hence verify that $F'(\alpha) = 0$, where α is any one of the roots of equation (A).

Differentiate using the quotient rule:

$$F'(x) = \frac{6x^2(3x^2 - 5) - 6x(2x^3 - 3)}{(3x^2 - 5)^2}.$$

Factorise the numerator:

$$F'(x) = \frac{6x\big(x(3x^2-5)-(2x^3-3)\big)}{(3x^2-5)^2}.$$

Inside the bracket,

$$x(3x^2-5)-(2x^3-3) = 3x^3-5x-2x^3+3 = x^3-5x+3.$$

So

$$F'(x) = \frac{6x(x^3-5x+3)}{(3x^2-5)^2}.$$

If α is a root of equation (A), then

$$\alpha^3-5\alpha+3=0.$$

Substituting $x=\alpha$ into the expression for $F'(x)$ gives

$$F'(\alpha) = \frac{6\alpha(\alpha^3-5\alpha+3)}{(3\alpha^2-5)^2} = 0.$$

$$F'(x) = \frac{6x(x^3-5x+3)}{(3x^2-5)^2}, \qquad F'(\alpha)=0$$

for any root α of (A).

(iii) Use the Newton–Raphson method to find the root of equation (A) which is close to 2. Write down sufficient approximations to find the root correct to 4 decimal places.

Using the formula from part (i), start with $x_1=2$. Then

$$x_2 = \frac{2(2)^3-3}{3(2)^2-5} = \frac{16-3}{12-5} = \frac{13}{7} = 1.857142857\ldots$$

so

$$x_2 \approx 1.85714.$$

Next,

$$x_3 = \frac{2x_2^3-3}{3x_2^2-5} \approx 1.83478735,$$

so

$$x_3 \approx 1.83479.$$

Next,

$$x_4 \approx 1.83424350,$$

so

$$x_4 \approx 1.83424.$$

Next,

$$x_5 \approx 1.83424318.$$

These agree to 4 decimal places, so the required root is

$$\alpha \approx 1.8342$$

(to 4 d.p.)

Question 7 (Jan 2013, Q8)

Worked Solution

It is required to solve

$$\ln(x-1) - x + 3 = 0.$$

There are two roots, α and β, where $1.1 < \alpha < 1.2$ and $4.1 < \beta < 4.2$.

(i)(a) The root β can be found using the iterative formula

$$x_{n+1} = \ln(x_n - 1) + 3.$$

Using this formula with $x_1 = 4.15$, find β correct to 3 decimal places.

Using the iteration:

$$x_2 = \ln(4.15 - 1) + 3 = \ln(3.15) + 3 \approx 4.1474.$$

Then

$$x_3 = \ln(4.1474 - 1) + 3 \approx \ln(3.1474) + 3 \approx 4.1466.$$

Then

$$x_4 = \ln(4.1466 - 1) + 3 \approx 4.1463.$$

Then

$$x_5 \approx 4.1462.$$

So the values are converging to

$$\beta \approx 4.146$$

(to 3 d.p.)

(i)(b) Explain with the aid of a sketch why this iterative formula will not converge to α, whatever initial value is taken.

Let

$$g(x) = \ln(x-1) + 3.$$

The iteration is $x_{n+1} = g(x_n)$, so graphically we compare $y = g(x)$ with $y = x$. The root α is the lower intersection point of these two graphs.

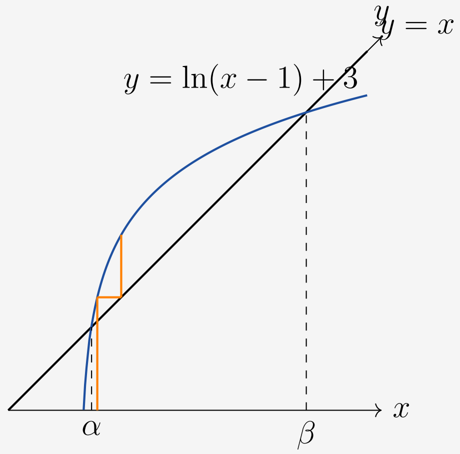
Near α, the curve $y = g(x)$ is steeper than the line $y = x$ because

$$g'(x) = \frac{1}{x-1},$$

and for x near α (where $1.1 < \alpha < 1.2$), we have

$$g'(\alpha) = \frac{1}{\alpha - 1} > \frac{1}{0.2} = 5.$$

So the staircase construction moves away from the lower fixed point and heads towards the upper root instead.



The staircase diagram moves away from the lower fixed point α, so this iteration does not converge to α.

(ii)(a) Show that the Newton–Raphson iterative formula can be written in the form

$$x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n}.$$

Take

$$f(x) = \ln(x-1) - x + 3.$$

Then

$$f'(x) = \frac{1}{x-1} - 1 = \frac{1-(x-1)}{x-1} = \frac{2-x}{x-1}.$$

Now Newton–Raphson gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\ln(x_n-1) - x_n + 3}{(2-x_n)/(x_n-1)}.$$

So

$$x_{n+1} = x_n - \frac{(x_n-1)(\ln(x_n-1) - x_n + 3)}{2-x_n}.$$

Write over a common denominator:

$$x_{n+1} = \frac{x_n(2-x_n) - (x_n-1)(\ln(x_n-1) - x_n + 3)}{2-x_n}.$$

Expanding the numerator,

$$= \frac{2x_n - x_n^2 - (x_n-1)\ln(x_n-1) + x_n(x_n-1) - 3(x_n-1)}{2-x_n}.$$

Now simplify:

$$2x_n - x_n^2 + x_n^2 - x_n - 3x_n + 3 = 3 - 2x_n.$$

Hence

$$x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n}.$$

$$\boxed{x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n}}$$

(ii)(b) Use this formula with $x_1 = 1.2$ to find α correct to 3 decimal places.

Using

$$x_{n+1} = \frac{3 - 2x_n - (x_n - 1)\ln(x_n - 1)}{2 - x_n},$$

start with $x_1 = 1.2$. Then

$$x_2 = \frac{3 - 2(1.2) - (0.2)\ln(0.2)}{2 - 1.2} = \frac{0.6 - 0.2(-1.6094379)}{0.8} \approx 1.1523595.$$

So

$$x_2 \approx 1.15236.$$

Next,

$$x_3 \approx 1.1584478,$$

so

$$x_3 \approx 1.15845.$$

Next,

$$x_4 \approx 1.1585943,$$

so

$$x_4 \approx 1.15859.$$

The values have stabilised, so

$$\boxed{\alpha \approx 1.159}$$

(to 3 d.p.)

Question 9 (Jun 2015, Q6i,iii,v)

Worked Solution

It is given that

$$3x^3 + 5x^2 - x - 1 = 0$$

has three roots, one of which is positive.

(i) Show that the Newton–Raphson iterative formula for finding this root can be written

$$x_{n+1} = \frac{6x_n^3 + 5x_n^2 + 1}{9x_n^2 + 10x_n - 1}.$$

Let

$$f(x) = 3x^3 + 5x^2 - x - 1.$$

Then

$$f'(x) = 9x^2 + 10x - 1.$$

Using Newton–Raphson,

$$x_{n+1} = x_n - \frac{3x_n^3 + 5x_n^2 - x_n - 1}{9x_n^2 + 10x_n - 1}.$$

Write as a single fraction:

$$x_{n+1} = \frac{x_n(9x_n^2 + 10x_n - 1) - (3x_n^3 + 5x_n^2 - x_n - 1)}{9x_n^2 + 10x_n - 1}.$$

Simplifying the numerator,

$$9x_n^3 + 10x_n^2 - x_n - 3x_n^3 - 5x_n^2 + x_n + 1 = 6x_n^3 + 5x_n^2 + 1.$$

Hence

$$x_{n+1} = \frac{6x_n^3 + 5x_n^2 + 1}{9x_n^2 + 10x_n - 1}.$$

$$\boxed{x_{n+1} = \frac{6x_n^3 + 5x_n^2 + 1}{9x_n^2 + 10x_n - 1}}$$

(iii) Apply the iterative formula in part (i) when the initial value is $x_1 = -1$. Describe the behaviour of the iterative sequence, illustrating your answer by the graph given in the Printed Answer Book.

Using $x_1 = -1$,

$$x_2 = \frac{6(-1)^3 + 5(-1)^2 + 1}{9(-1)^2 + 10(-1) - 1} = \frac{-6 + 5 + 1}{9 - 10 - 1} = \frac{0}{-2} = 0.$$

Then

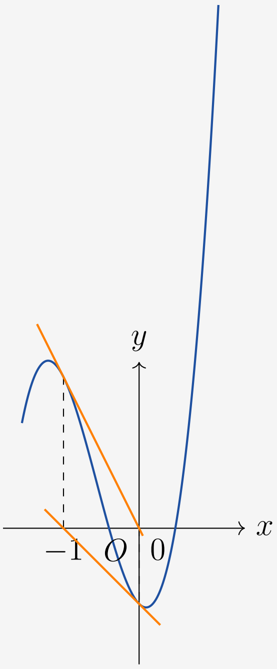
$$x_3 = \frac{6(0)^3 + 5(0)^2 + 1}{9(0)^2 + 10(0) - 1} = \frac{1}{-1} = -1.$$

So the sequence continues

$$-1,\ 0,\ -1,\ 0,\ -1,\ 0, \dots$$

It alternates between -1 and 0, so it does not converge.

A sketch interpretation is that the tangent from $x = -1$ hits the x-axis at 0, and the tangent from $x = 0$ hits the x-axis back at -1.



The sequence alternates between -1 and 0, so it does not converge.

(v) Find the value of the positive root correct to 5 decimal places.

Continue the Newton–Raphson process using a starting value that converges to the positive root. Since the positive root lies between 0 and 0.5, a suitable choice is $x_1 = 0.5$.

Using

$$x_{n+1} = \frac{6x_n^3 + 5x_n^2 + 1}{9x_n^2 + 10x_n - 1},$$

we get

$$x_2 = \frac{6(0.5)^3 + 5(0.5)^2 + 1}{9(0.5)^2 + 10(0.5) - 1} = \frac{0.75 + 1.25 + 1}{2.25 + 5 - 1} = \frac{3}{6.25} = 0.48.$$

Then

$$x_3 \approx 0.4793571,$$

and

$$x_4 \approx 0.4793565.$$

So the positive root is

$$\boxed{0.47936}$$

(to 5 d.p.)

End of Worked Solutions