Question 1

Worked Solution

$f(x) = \ln(2x - 5) + 2x^2 - 30$, $x > 2.5$

Part (a): Show $f(x) = 0$ has a root α in $[3.5, 4]$

$$f(3.5) = \ln(2) + 2(12.25) - 30 = 0.693\ldots + 24.5 - 30 = -4.81 < 0$$
$$f(4) = \ln(3) + 2(16) - 30 = 1.099\ldots + 32 - 30 = 3.10 > 0$$

There is a change of sign, and $f(x)$ is continuous on $[3.5, 4]$, so there is a root α in the interval $[3.5, 4]$.

Sign change confirmed; root in [3.5, 4].

Part (b): Apply Newton-Raphson once from $x_0 = 4$

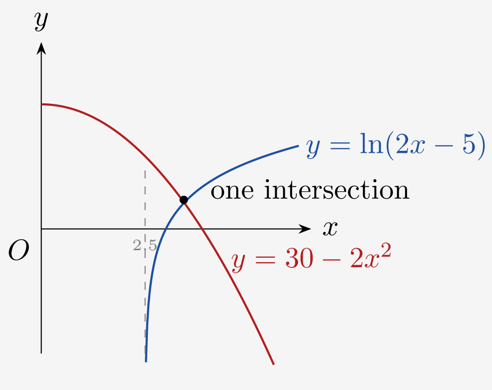
Using $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ with $f(4) = 3.099$ and $f'(4) = 16.67$:

$$x_1 = 4 - \frac{3.099}{16.67} = 4 - 0.1859\ldots = 3.814\ldots$$

Second approximation $x_1 = 3.81$ (3 s.f.)

Part (c): Show α is the only root of $f(x) = 0$

Rewrite $f(x) = 0$ as $\ln(2x - 5) = 30 - 2x^2$, i.e. find where $y = \ln(2x - 5)$ meets $y = 30 - 2x^2$.



The curve $y = \ln(2x - 5)$ is strictly increasing for $x > 2.5$ (starting at $-\infty$), while $y = 30 - 2x^2$ is strictly decreasing for $x > 0$. A strictly increasing function meets a strictly decreasing function at most once, so there is exactly one intersection, meaning $f(x) = 0$ has exactly one root.

$y = \ln(2x - 5)$ is increasing; $y = 30 - 2x^2$ is decreasing. They intersect exactly once, so α is the only root.

Question 2

Worked Solution

$\mathrm{f}(x) = x^3 - \dfrac{5}{2x^{3/2}} + 2x - 3,\ x > 0$

Part (a): Show root α in $[1.1, 1.5]$

$$\mathrm{f}(1.1) = (1.1)^3 - \frac{5}{2(1.1)^{3/2}} + 2(1.1) - 3 = 1.331 - 2.165 + 2.2 - 3 = -1.634\ldots < 0$$

$$\mathrm{f}(1.5) = (1.5)^3 - \frac{5}{2(1.5)^{3/2}} + 2(1.5) - 3 = 3.375 - 1.361 + 3 - 3 = 2.014\ldots > 0$$

Sign change and f is continuous on $[1.1, 1.5]$, so there is a root α in $[1.1, 1.5]$.

> $\mathrm{f}(1.1) = -1.634 < 0$, $\mathrm{f}(1.5) = 2.014 > 0$; sign change $\Rightarrow$ root in $[1.1, 1.5]$.

Part (b): Find $\mathrm{f}'(x)$

Write $\mathrm{f}(x) = x^3 - \frac{5}{2}x^{-3/2} + 2x - 3$.

$$\mathrm{f}'(x) = 3x^2 + \frac{15}{4}x^{-5/2} + 2$$

> $\mathrm{f}'(x) = 3x^2 + \dfrac{15}{4}x^{-5/2} + 2$

Part (c): Apply Newton-Raphson once from $x_0 = 1.1$

$\mathrm{f}(1.1) = -1.6360\ldots \qquad \mathrm{f}'(1.1) = 3(1.21) + \dfrac{15}{4}(1.1)^{-5/2} + 2 = 3.63 + 2.955 + 2 = 8.585\ldots$

$$x_1 = 1.1 - \frac{-1.6360}{8.585} = 1.1 + 0.1906\ldots = 1.2906\ldots$$

> Second approximation $= 1.291$ (3 d.p.)

Question 3

Worked Solution

$\mathrm{f}(x) = x^2 + \dfrac{3}{4\sqrt{x}} - 3x - 7$, $x > 0$. Root in $[3, 5]$; first approximation $x_0 = 4$.

Find $\mathrm{f}'(x)$

Write $\mathrm{f}(x) = x^2 + \frac{3}{4}x^{-1/2} - 3x - 7$.

$$\mathrm{f}'(x) = 2x - \frac{3}{8}x^{-3/2} - 3$$

Apply Newton-Raphson once

$$\mathrm{f}(4) = 16 + \frac{3}{4\sqrt{4}} - 12 - 7 = 16 + \frac{3}{8} - 12 - 7 = -2.625$$

$$\mathrm{f}'(4) = 8 - \frac{3}{8(4)^{3/2}} - 3 = 8 - \frac{3}{64} - 3 = 4.953125$$

$$x_1 = 4 - \frac{-2.625}{4.953125} = 4 + 0.5300\ldots = 4.5300\ldots$$

Second approximation $= 4.53$ (2 d.p.)

Question 4

Worked Solution

$\mathrm{f}(x) = x + \tan\left(\frac{1}{2}x\right)$, $\pi < x < \frac{3\pi}{2}$. Single root α.

Part (a): Show $\alpha \in [3.6, 3.7]$

$$\mathrm{f}(3.6) = 3.6 + \tan(1.8) = 3.6 + (-4.286\ldots) = -0.686\ldots < 0$$
$$\mathrm{f}(3.7) = 3.7 + \tan(1.85) = 3.7 + (-3.489\ldots) = 0.211\ldots > 0$$

Sign change, f continuous on $[3.6, 3.7]$ (no asymptote in this subinterval), so $\alpha \in [3.6, 3.7]$.

$\mathrm{f}(3.6) = -0.686 < 0$, $\mathrm{f}(3.7) = 0.211 > 0$; sign change $\Rightarrow \alpha \in [3.6, 3.7]$.

Part (b): Find $\mathrm{f}'(x)$

$$\mathrm{f}'(x) = 1 + \frac{1}{2}\sec^2\left(\frac{1}{2}x\right)$$

$\mathrm{f}'(x) = 1 + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

Part (c): Apply Newton-Raphson once from $x_0 = 3.7$

$\mathrm{f}(3.7) = 0.2119\ldots \qquad \mathrm{f}'(3.7) = 1 + \frac{1}{2}\sec^2(1.85) = 1 + \frac{1}{2}(1 + \tan^2(1.85)) = 7.583\ldots$

$$x_1 = 3.7 - \frac{0.2119}{7.583} = 3.7 - 0.02794\ldots = 3.6721\ldots$$

Second approximation $= 3.672$ (3 d.p.)

Question 5

Worked Solution

$f(x) = 3\sqrt{x} + \dfrac{18}{\sqrt{x}} - 20$

Part (a): Show root α in $[1.1, 1.2]$

$$f(1.1) = 3\sqrt{1.1} + \frac{18}{\sqrt{1.1}} - 20 = 3.1464 + 17.1534 - 20 = 0.3088\ldots > 0$$
$$f(1.2) = 3\sqrt{1.2} + \frac{18}{\sqrt{1.2}} - 20 = 3.2863 + 16.4317 - 20 = -0.2820\ldots < 0$$

Sign change, f is continuous on $[1.1, 1.2]$, so there is a root α in $[1.1, 1.2]$.

Sign change confirmed; root in $[1.1, 1.2]$.

Part (b): Find $f'(x)$

Write $f(x) = 3x^{1/2} + 18x^{-1/2} - 20$.

$$f'(x) = \frac{3}{2}x^{-1/2} - 9x^{-3/2}$$

$f'(x) = \dfrac{3}{2}x^{-1/2} - 9x^{-3/2}$

Part (c): Apply Newton-Raphson once from $x_0 = 1.1$

$f(1.1) = 0.3088\ldots \qquad f'(1.1) = \dfrac{3}{2}(1.1)^{-1/2} - 9(1.1)^{-3/2} = 1.4289 - 7.7994 = -6.3706\ldots$

$$x_1 = 1.1 - \frac{0.3088}{-6.3706} = 1.1 + 0.04847\ldots = 1.1485\ldots$$

Second approximation $= 1.15$ (3 s.f.)

Question 6

Worked Solution

$f(x) = 2^x - 10x$

Part (a): Show $\alpha \in [5, 6]$

$$f(5) = 2^5 - 50 = 32 - 50 = -18 < 0$$
$$f(6) = 2^6 - 60 = 64 - 60 = 4 > 0$$

Sign change, f is continuous, so there is a root α in $[5, 6]$.

$f(5) = -18 < 0$, $f(6) = 4 > 0$; sign change $\Rightarrow \alpha \in [5, 6]$.

Part (b): Value of p

Differentiating $f(x) = 2^x - 10x$:

$$f'(x) = 2^x \ln 2 - 10$$

Comparing with $f'(x) = p \times 2^x - 10$:

$p = \ln 2$

Part (c): Apply Newton-Raphson once from $x_0 = 6$

$$f(6) = 4 \qquad f'(6) = 64 \ln 2 - 10 = 44.361\ldots - 10 = 34.361\ldots$$

$$x_1 = 6 - \frac{4}{34.361} = 6 - 0.1164\ldots = 5.884\ldots$$

Second approximation $= 5.88$ (3 s.f.)

Part (d): x-coordinate of minimum turning point Q

At the minimum, $f'(x) = 0$:

$$2^x \ln 2 - 10 = 0 \implies 2^x = \frac{10}{\ln 2} \implies x = \log_2\left(\frac{10}{\ln 2}\right) = \frac{\ln\left(\frac{10}{\ln 2}\right)}{\ln 2}$$

$$x = \frac{\ln(10/0.6931\ldots)}{0.6931\ldots} = \frac{\ln(14.427\ldots)}{0.6931\ldots} = \frac{2.668\ldots}{0.6931\ldots} = 3.851\ldots$$

x-coordinate of Q $= 3.85$ (3 s.f.)

Question 7

Worked Solution

$f(x) = 8\sin\left(\frac{1}{2}x\right) - 3x + 9,\ x > 0$ (radians)

Part (a): x-coordinate of local maximum P

$f'(x) = 4\cos\left(\frac{1}{2}x\right) - 3$

Setting $f'(x) = 0$:

$$4\cos\left(\frac{x}{2}\right) = 3 \implies \cos\left(\frac{x}{2}\right) = \frac{3}{4} \implies \frac{x}{2} = \arccos\left(\frac{3}{4}\right) \implies x = 2\arccos\left(\frac{3}{4}\right)$$

$$x = 2 \times 0.7227\ldots = 1.4455\ldots$$

To confirm this is a maximum: $f''(x) = -2\sin\left(\frac{x}{2}\right) < 0$ at $x \approx 1.445$. ✓

x-coordinate of P $= 14.0$ (3 s.f.)

Part (b): Why $\alpha \in [4, 5]$

$f(4) = 4.274 > 0$ and $f(5) = -1.212 < 0$. There is a change of sign, and f is continuous on $[4, 5]$, so there must be a root α in the interval $[4, 5]$.

Part (c): Apply Newton-Raphson once from $x_0 = 5$

$f(5) = -1.212 \qquad f'(5) = 4\cos(2.5) - 3 = 4(-0.8011\ldots) - 3 = -3.2044 - 3 = -6.204\ldots$

$$x_1 = 5 - \frac{-1.212}{-6.204} = 5 - 0.1954\ldots = 4.805\ldots$$

Second approximation $= 4.80$ (3 s.f.)

End of Worked Solutions