Question 1

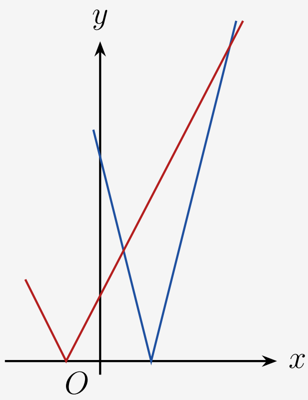
Worked Solution

Solve the inequality $|4x-3| < |2x+1|$.

To avoid rogue solutions, sketch the graphs

$$y = |4x-3| \qquad \text{and} \qquad y = |2x+1|$$

on the same axes.



The first graph has vertex at

$$4x - 3 = 0 \implies x = \frac{3}{4},$$

so its vertex is $\left(\frac{3}{4}, 0\right)$.

The second graph has vertex at

$$2x + 1 = 0 \implies x = -\frac{1}{2},$$

so its vertex is $\left(-\frac{1}{2}, 0\right)$.

The critical values come from the intersections of the two graphs.

Right-hand branch of $y = |4x-3|$:

$$4x - 3 = |2x+1|.$$

For the relevant branch here, $2x + 1 \geq 0$, so

$$4x - 3 = 2x + 1 \implies 2x = 4 \implies x = 2.$$

Left-hand branch of $y = |4x-3|$:

$$3 - 4x = |2x+1|.$$

For the relevant branch here, $2x + 1 \geq 0$, so

$$3 - 4x = 2x + 1 \implies 2 = 6x \implies x = \frac{1}{3}.$$

So the graphs intersect at $x = \frac{1}{3}$ and $x = 2$.

From the sketch, $|4x - 3| < |2x + 1|$ where the graph of $y = |4x - 3|$ lies below the graph of $y = |2x + 1|$, namely between the intersection points.

$$\boxed{\frac{1}{3} < x < 2}$$

Question 2

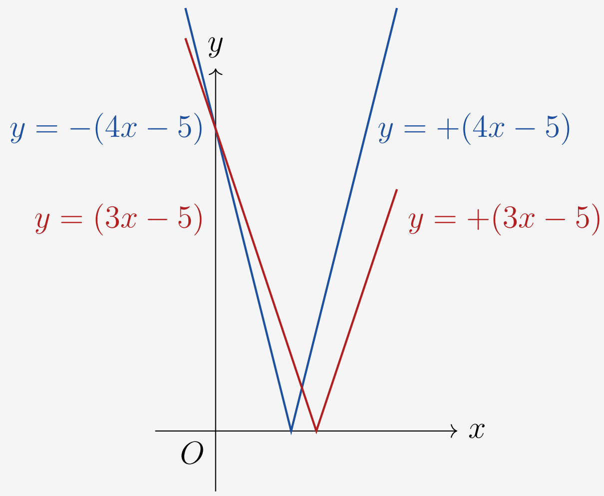
Worked Solution

Find the exact solutions of $|4x-5| = |3x-5|$.

Sketch the graphs

$$y = |4x-5| \qquad \text{and} \qquad y = |3x-5|$$

on the same axes. The solutions are the x-coordinates of their intersections.



The vertices are at

$$4x-5=0 \implies x=\frac{5}{4}, \qquad 3x-5=0 \implies x=\frac{5}{3}.$$

Now solve the simultaneous equations for the relevant branches.

First intersection: Both expressions are negative for small x, so

$$5-4x = 5-3x \implies x=0.$$

Second intersection: Between $\frac{5}{4}$ and $\frac{5}{3}$, the signs are different, so

$$4x-5 = 5-3x \implies 7x=10 \implies x=\frac{10}{7}.$$

There are no further intersections.

$$\boxed{x = 0 \text{ or } x = \frac{10}{7}}$$

Question 3

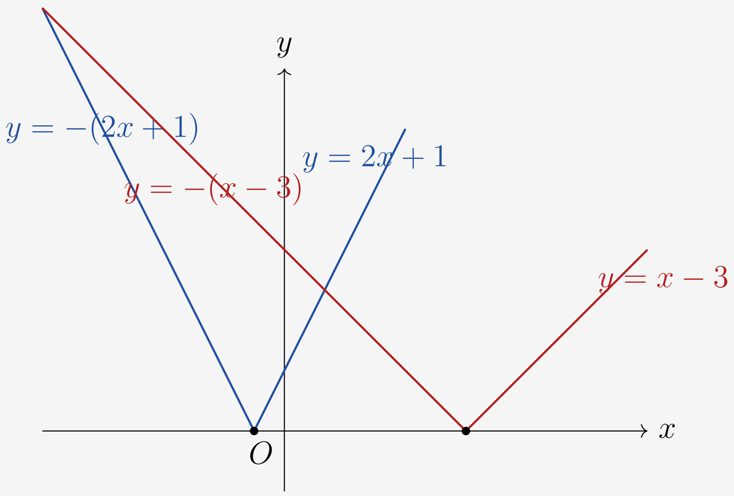
Worked Solution

(i) Solve the inequality $|2x+1| \le |x-3|$.

Sketch the graphs

$$y = |2x+1| \qquad \text{and} \qquad y = |x-3|$$

on the same axes.



Their vertices are at

$$2x+1=0 \implies x=-\frac{1}{2}, \qquad x-3=0 \implies x=3.$$

The critical values come from the intersections.

Left intersection: For $x < -\frac{1}{2}$, both expressions are negative, so

$$-(2x+1) = -(x-3)$$

$$-2x-1 = -x+3$$

$$-x = 4 \implies x = -4.$$

Right intersection: For $-\frac{1}{2} \le x < 3$, we have $2x+1 \ge 0$ and $x-3<0$, so

$$2x+1 = -(x-3) = 3-x$$

$$3x = 2 \implies x = \frac{2}{3}.$$

So the graphs intersect at $x=-4$ and $x=\frac{2}{3}$.

From the sketch, $|2x+1| \le |x-3|$ where the graph of $y=|2x+1|$ is below or equal to the graph of $y=|x-3|$, which is between these two values.

$$\boxed{-4 \le x \le \frac{2}{3}}$$

(ii) Given that x satisfies $|2x+1| \le |x-3|$, find the greatest possible value of $|x+2|$.

From part (i),

$$-4 \le x \le \frac{2}{3}.$$

So $x+2$ ranges from

$$-4+2=-2 \quad \text{to} \quad \frac{2}{3}+2=\frac{8}{3}.$$

Hence

$$|x+2|$$

is greatest at an endpoint, and we compare

$$|-2|=2, \qquad \left|\frac{8}{3}\right|=\frac{8}{3}.$$

$$\boxed{\max|x+2|=\frac{8}{3}}$$

Question 4

Worked Solution

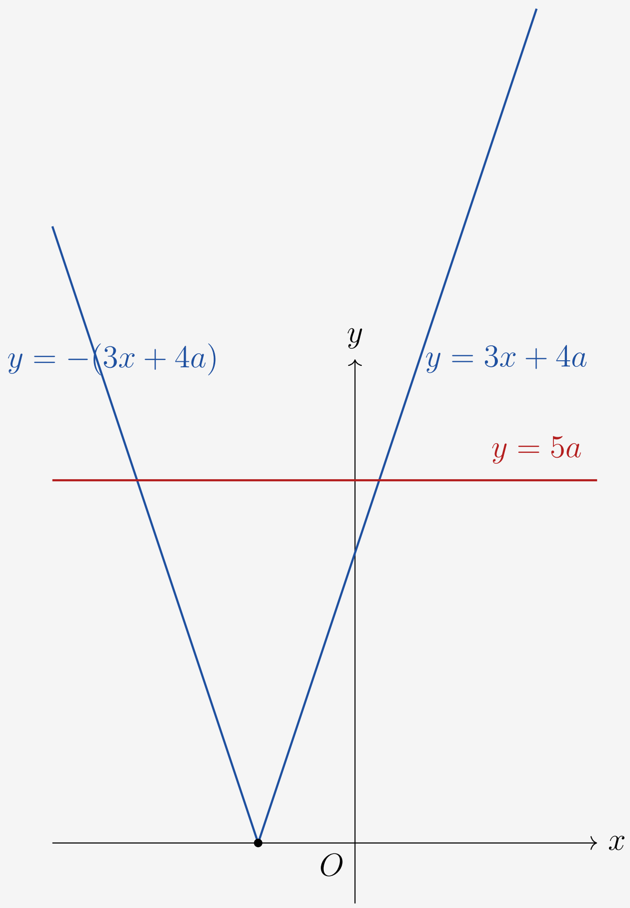
Solve the equation $|3x + 4a| = 5a$, where a is a positive constant.

Sketch the graph

$$y = |3x + 4a|$$

and the horizontal line

$$y = 5a.$$



The solutions are the x-coordinates of the points of intersection.

Since

$$|3x + 4a| = 5a,$$

there are two branches to consider.

Right-hand branch:

$$3x + 4a = 5a \implies 3x = a \implies x = \frac{a}{3}.$$

Left-hand branch:

$$-(3x + 4a) = 5a \implies -3x - 4a = 5a \implies -3x = 9a \implies x = -3a.$$

$$\boxed{x = -3a \text{ or } x = \frac{a}{3}}$$

Question 5

Worked Solution

The functions are

$$f(x) = |x|, \qquad g(x) = 3x + 5, \qquad h(x) = gg(x).$$

First find $h(x)$:

$$h(x) = g(g(x)) = g(3x+5) = 3(3x+5) + 5 = 9x + 20.$$

(i) Solve $g(x+2) = f(-12)$.

Since

$$f(-12) = |-12| = 12,$$

we solve

$$g(x+2) = 12.$$

Now

$$g(x+2) = 3(x+2) + 5 = 3x + 11,$$

so

$$3x + 11 = 12 \implies 3x = 1 \implies x = \frac{1}{3}.$$

$$\boxed{x = \frac{1}{3}}$$

(ii) Find $h^{-1}(x)$.

Let

$$y = 9x + 20.$$

Rearrange to make x the subject:

$$9x = y - 20 \implies x = \frac{y-20}{9}.$$

Hence

$$h^{-1}(x) = \frac{x-20}{9}.$$

$$\boxed{h^{-1}(x) = \frac{x-20}{9}}$$

(iii) Determine the values of x for which $x + f(x) = 0$.

We need to solve

$$x + |x| = 0.$$

If $x \geq 0$, then $|x| = x$, so

$$x + x = 0 \implies 2x = 0 \implies x = 0.$$

If $x < 0$, then $|x| = -x$, so

$$x + (-x) = 0,$$

which is true for every $x < 0$.

Hence all non-positive values of x satisfy the equation.

$$\boxed{x \leq 0}$$

Question 6

Worked Solution

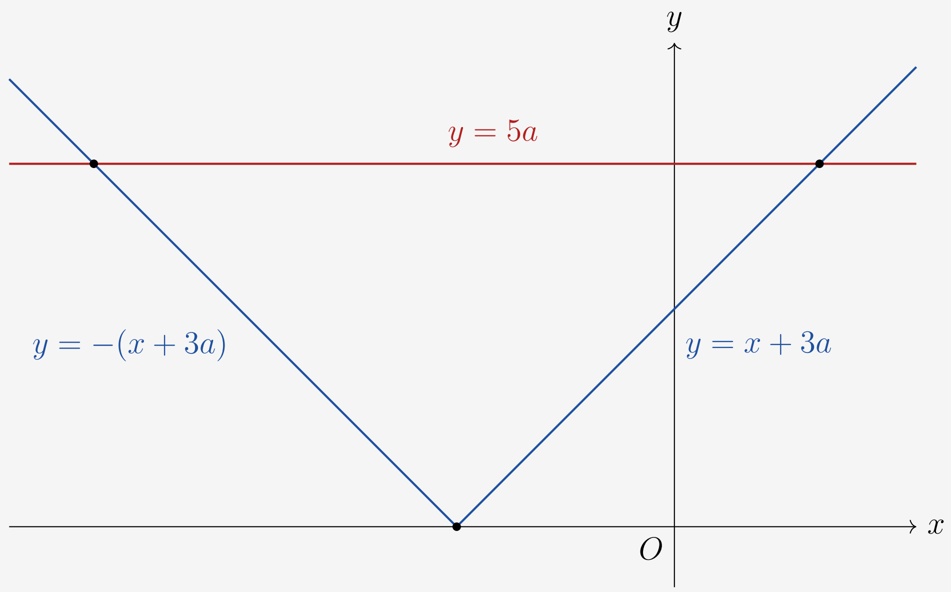
It is given that $|x+3a| = 5a$, where a is a positive constant. Find, in terms of a, the possible values of

$$|x+7a| - |x-7a|.$$

First solve

$$|x+3a| = 5a.$$

Using the graph of $y = |x+3a|$ and the line $y = 5a$, or equivalently solving each branch,



$$x+3a = 5a \implies x = 2a,$$

and

$$-(x+3a) = 5a \implies -x = 8a \implies x = -8a.$$

So there are two possible values of x.

Case 1: $x = 2a$

$$|x+7a| - |x-7a| = |2a+7a| - |2a-7a| = |9a| - |-5a| = 9a - 5a = 4a.$$

Case 2: $x = -8a$

$$|x+7a| - |x-7a| = |-8a+7a| - |-8a-7a| = |-a| - |-15a| = a - 15a = -14a.$$

$$\boxed{4a \text{ or } -14a}$$

Question 7

Worked Solution

The functions are

$$f(x) = |2x + a| + 3a, \qquad g(x) = 5x - 4a,$$

where $a > 0$.

(i) State the range of f and the range of g.

Since $|2x + a| \geq 0$,

$$f(x) = |2x + a| + 3a \geq 3a.$$

The minimum value $3a$ occurs when $2x + a = 0$, i.e. when $x = -\frac{a}{2}$. So the range of f is

$$[3a, \infty).$$

The function $g(x) = 5x - 4a$ is linear with no restriction on x, so its range is all real numbers.

$$\boxed{\text{Range of } f = [3a, \infty), \qquad \text{Range of } g = \mathbb{R}}$$

(ii) State why f has no inverse, and find an expression for $g^{-1}(x)$.

The graph of f is a V-shape, so it is not one-to-one: different values of x can give the same value of $f(x)$. Therefore f has no inverse function on all real numbers.

To find $g^{-1}(x)$, let

$$y = 5x - 4a.$$

Then

$$5x = y + 4a \implies x = \frac{y + 4a}{5}.$$

Hence

$$g^{-1}(x) = \frac{x + 4a}{5}.$$

$$\boxed{f \text{ has no inverse because it is not one-to-one}, \qquad g^{-1}(x) = \frac{x + 4a}{5}}$$

(iii) Solve for x the equation $gf(x) = 31a$.

First compose the functions:

$$gf(x) = g(f(x)) = 5f(x) - 4a = 5\big(|2x + a| + 3a\big) - 4a.$$

So

$$gf(x) = 5|2x + a| + 15a - 4a = 5|2x + a| + 11a.$$

Set this equal to $31a$:

$$5|2x+a|+11a=31a$$
$$5|2x+a|=20a$$
$$|2x+a|=4a.$$

Now sketch the graph $y=|2x+a|$ and the line $y=4a$. The solutions are the two intersection points.

Right-hand branch:

$$2x+a=4a \implies 2x=3a \implies x=\frac{3a}{2}.$$

Left-hand branch:

$$-(2x+a)=4a \implies -2x-a=4a \implies -2x=5a \implies x=-\frac{5a}{2}.$$

$$\boxed{x=-\frac{5a}{2} \text{ or } x=\frac{3a}{2}}$$

Question 8

Worked Solution

The given graph is $y = -\sin^{-1}(x-1)$.

(i) Give details of the pair of geometrical transformations which transforms the graph of $y = -\sin^{-1}(x-1)$ to the graph of $y = \sin^{-1} x$.

Starting with

$$y = -\sin^{-1}(x-1),$$

reflect in the x-axis to get

$$y = \sin^{-1}(x-1).$$

Then translate 1 unit to the left to get

$$y = \sin^{-1} x.$$

Reflect in the x-axis, then translate 1 unit left.

(ii) Sketch the graph of $y = |-\sin^{-1}(x-1)|$.

Since

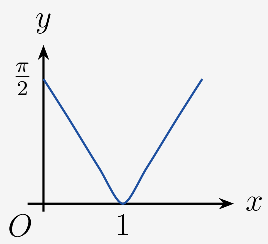
$$|-\sin^{-1}(x-1)| = |\sin^{-1}(x-1)|,$$

we keep the part of the graph above the x-axis unchanged and reflect the part below the x-axis in the x-axis.

The original graph crosses the x-axis at $x = 1$, and has endpoints

$$(0, \tfrac{\pi}{2}) \quad \text{and} \quad (2, -\tfrac{\pi}{2}).$$

So the new graph runs from $(0, \frac{\pi}{2})$ down to $(1, 0)$, then back up to $(2, \frac{\pi}{2})$.



(iii) Find the exact solutions of $|-\sin^{-1}(x-1)| = \frac{\pi}{3}$.

Using the sketch from part (ii), draw the horizontal line

$$y = \frac{\pi}{3}.$$

It meets the graph at two points.

Since

$$|-\sin^{-1}(x-1)| = \frac{\pi}{3} \quad \Longleftrightarrow \quad |\sin^{-1}(x-1)| = \frac{\pi}{3},$$

we have

$$\sin^{-1}(x-1) = \pm\frac{\pi}{3}.$$

Thus

$$x - 1 = \sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}.$$

So

$$x = 1 \pm \frac{\sqrt{3}}{2}.$$

$$\boxed{x = 1 - \frac{\sqrt{3}}{2} \text{ or } x = 1 + \frac{\sqrt{3}}{2}}$$

Question 9

Worked Solution

The functions are

$$f(x) = 3x - 2, \qquad g(x) = 3x + 7.$$

(i) Find the exact coordinates of the point at which the graph of $y = fg(x)$ meets the x-axis.

First compose the functions:

$$fg(x) = f(g(x)) = 3(3x + 7) - 2 = 9x + 19.$$

At the x-axis, $y = 0$, so

$$9x + 19 = 0 \implies x = -\frac{19}{9}.$$

$$\boxed{\left(-\frac{19}{9}, 0\right)}$$

(ii) Find the exact coordinates of the point at which the graph of $y = g(x)$ meets the graph of $y = g^{-1}(x)$.

First find the inverse of g:

$$y = 3x + 7 \implies x = \frac{y - 7}{3}.$$

Hence

$$g^{-1}(x) = \frac{x - 7}{3}.$$

Now solve

$$3x + 7 = \frac{x - 7}{3}.$$

Multiply by 3:

$$9x + 21 = x - 7$$

$$8x = -28 \implies x = -\frac{7}{2}.$$

Then

$$y = 3\left(-\frac{7}{2}\right) + 7 = -\frac{21}{2} + \frac{14}{2} = -\frac{7}{2}.$$

$$\boxed{\left(-\frac{7}{2}, -\frac{7}{2}\right)}$$

(iii) Find the exact coordinates of the point at which the graph of $y = |f(x)|$ meets the graph of $y = |g(x)|$.

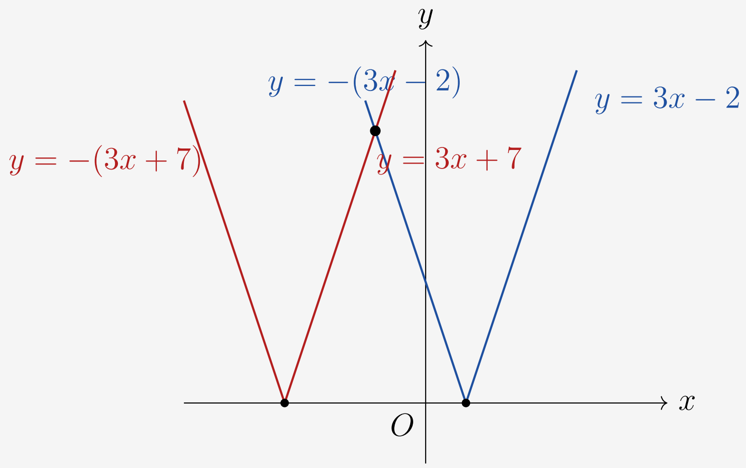
We need to solve

$$|3x - 2| = |3x + 7|.$$

Sketch the graphs

$$y = |3x - 2| \qquad \text{and} \qquad y = |3x + 7|$$

on the same axes.



Their vertices are at

$$\left(\frac{2}{3}, 0\right) \qquad \text{and} \qquad \left(-\frac{7}{3}, 0\right).$$

The intersection occurs where one inside expression is positive and the other negative, so solve

$$3x - 2 = -(3x + 7).$$

Then

$$3x - 2 = -3x - 7$$

$$6x = -5 \implies x = -\frac{5}{6}.$$

Now find the corresponding y-value:

$$y = \left|3\left(-\frac{5}{6}\right) - 2\right| = \left|-\frac{5}{2} - 2\right| = \left|-\frac{9}{2}\right| = \frac{9}{2}.$$

$$\boxed{\left(-\frac{5}{6}, \frac{9}{2}\right)}$$

End of Worked Solutions