Question 1

Worked Solution

(a) Sketch $y = |4x - 3|$ and state where it cuts or meets the axes.

The graph of $y = |4x - 3|$ is a V-shape.

Its vertex occurs when $4x - 3 = 0$, so

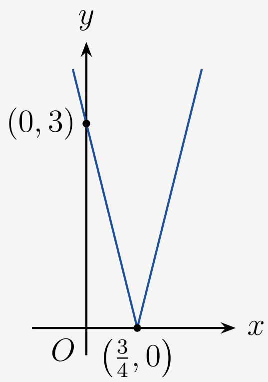
$$4x - 3 = 0 \implies x = \frac{3}{4}.$$

Hence the vertex is $\left(\frac{3}{4}, 0\right)$.

The y-intercept is found by putting $x = 0$:

$$y = |4(0) - 3| = 3,$$

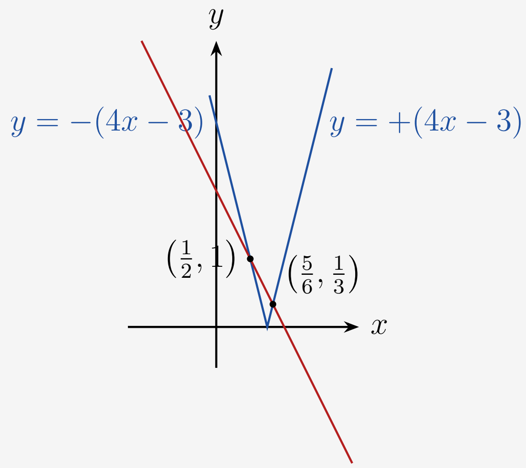
so the graph cuts the y-axis at $(0, 3)$.



The graph meets the x-axis at $\left(\frac{3}{4}, 0\right)$ and cuts the y-axis at $(0, 3)$.

(b) Solve $|4x - 3| > 2 - 2x$.

Use the graphs $y = |4x - 3|$ and $y = 2 - 2x$ on the same axes. First find their points of intersection.



Solve the simultaneous equations with each branch of the modulus graph.

For $x \geq \dfrac{3}{4}$, we have $|4x - 3| = 4x - 3$, so

$$4x - 3 = 2 - 2x$$
$$6x = 5$$
$$x = \frac{5}{6}.$$

For $x < \dfrac{3}{4}$, we have $|4x - 3| = 3 - 4x$, so

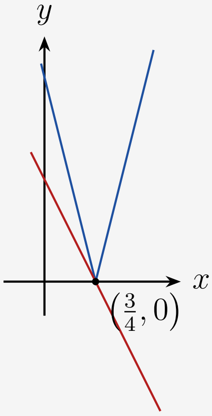
$$3 - 4x = 2 - 2x$$
$$1 = 2x$$
$$x = \frac{1}{2}.$$

From the sketch, $|4x - 3|$ is *above* the line $y = 2 - 2x$ to the left of $x = \frac{1}{2}$ and to the right of $x = \frac{5}{6}$.

$$\boxed{x < \frac{1}{2} \quad \text{or} \quad x > \frac{5}{6}}$$

(c) Solve $|4x - 3| > \dfrac{3}{2} - 2x$**.**

Again sketch $y = |4x - 3|$ and $y = \dfrac{3}{2} - 2x$ on the same axes. The line passes through the vertex of the V-shape.



Check the intersection algebraically.

For $x \geq \dfrac{3}{4}$:

$$4x - 3 = \frac{3}{2} - 2x \implies 6x = \frac{9}{2} \implies x = \frac{3}{4}.$$

For $x < \dfrac{3}{4}$:

$$3 - 4x = \frac{3}{2} - 2x \implies \frac{3}{2} = 2x \implies x = \frac{3}{4}.$$

So the line touches the modulus graph only at $x = \frac{3}{4}$. Therefore $|4x - 3|$ is greater than $\dfrac{3}{2} - 2x$ for every value of x except $x = \frac{3}{4}$.

$$\boxed{x \in \mathbb{R},\ x \neq \frac{3}{4}}$$

Question 2

Worked Solution

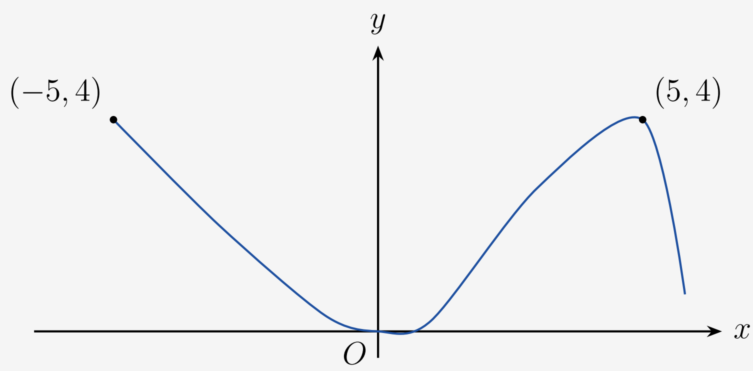
The original curve $y = f(x)$ passes through $O(0,0)$, $A(5,4)$ and $B(-5,-4)$.

(a) Sketch $y = |f(x)|$**.**

Taking the modulus of the whole function reflects every part of the graph below the x-axis in the x-axis.

So:

$$A(5,4) \text{ stays at } (5,4), \qquad B(-5,-4) \text{ moves to } (-5,4).$$

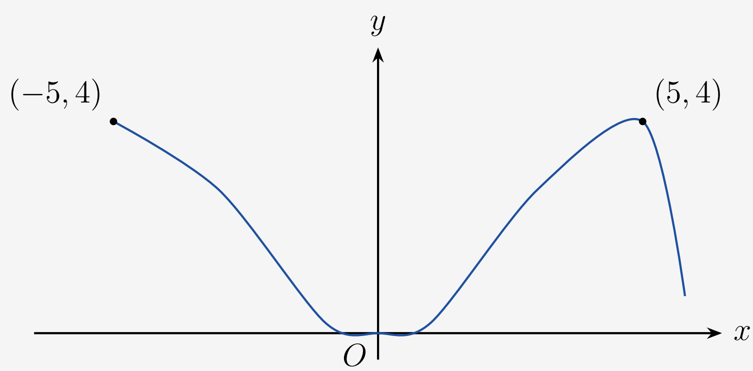


For $y = |f(x)|$, the part below the x-axis is reflected above it. The corresponding points are $(5,4)$ and $(-5,4)$.

(b) Sketch $y = f(|x|)$**.**

Replacing x by $|x|$ copies the right-hand side of the graph into the left-hand side by reflection in the y-axis.

So the right-hand point $(5,4)$ is mirrored to $(-5,4)$.



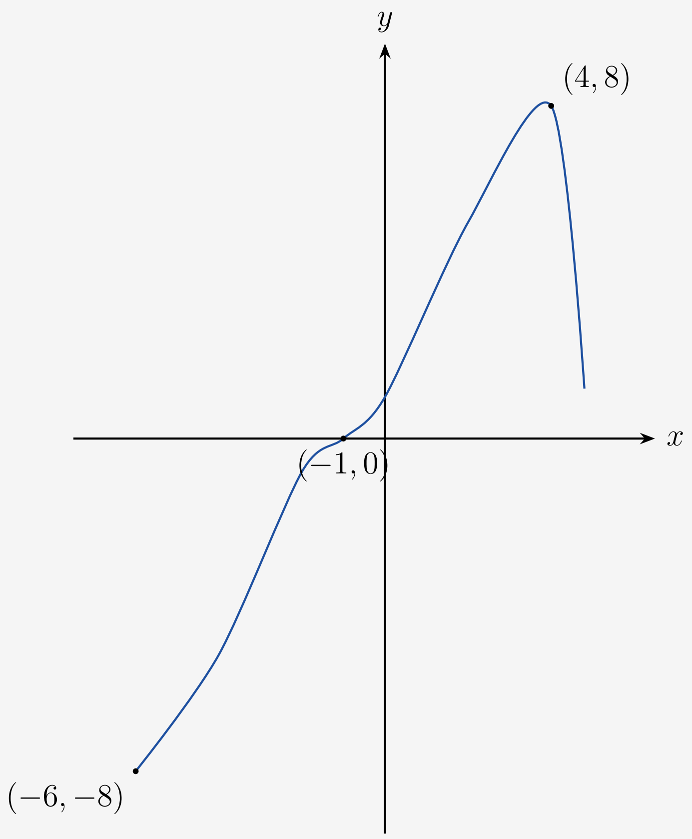
For $y = f(|x|)$, the right-hand side is reflected in the y-axis. The corresponding points are $(5,4)$ and $(-5,4)$.

(c) Sketch $y = 2f(x + 1)$.

The transformation $x \mapsto x + 1$ shifts the graph 1 unit to the left. Then multiplying by 2 stretches it parallel to the y-axis by scale factor 2.

Therefore:

$$A(5,4) \mapsto (4,8), \qquad B(-5,-4) \mapsto (-6,-8), \qquad O(0,0) \mapsto (-1,0).$$



For $y = 2f(x + 1)$, the corresponding points are $(4, 8)$ and $(-6, -8)$.

Question 3

Worked Solution

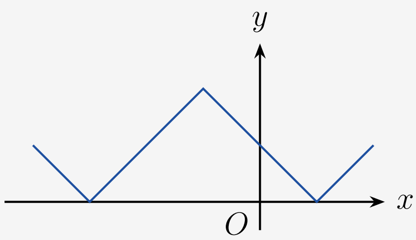
Given

$$f(x) = 2 - |x + 1|.$$

(a) Sketch $y = |f(x)|$.

The graph of $y = f(x)$ is an inverted V with vertex at $(-1, 2)$. Taking the modulus reflects the parts below the x-axis above the x-axis.

The x-intercepts of $f(x)$ are at $x = -3$ and $x = 1$, so these become sharp points on the new graph.

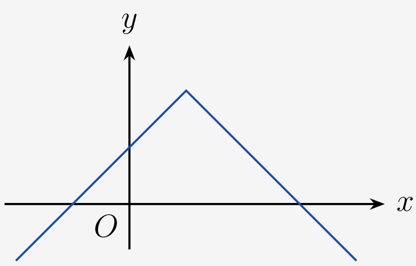


The graph is reflected in the x-axis wherever $f(x) < 0$, giving vertices at $(-3, 0)$, $(-1, 2)$ and $(1, 0)$.

(b) Sketch $y = f(-x)$.

Replacing x by $-x$ reflects the graph of $y = f(x)$ in the y-axis.

So the vertex $(-1, 2)$ moves to $(1, 2)$, and the intercepts $(-3, 0)$ and $(1, 0)$ move to $(3, 0)$ and $(-1, 0)$ respectively.



$y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

(c) Find the coordinates of P, Q and R.

Since

$$f(x) = 2 - |x + 1|,$$

the maximum occurs when $x + 1 = 0$, that is when $x = -1$.

So

$$P = (-1, 2).$$

For the y-intercept, put $x = 0$:

$$Q = (0, f(0)) = (0, 2 - |1|) = (0, 1).$$

For the right-hand x-intercept R, solve

$$2 - |x + 1| = 0 \implies |x + 1| = 2.$$

Hence

$$x + 1 = \pm 2 \implies x = 1 \text{ or } x = -3.$$

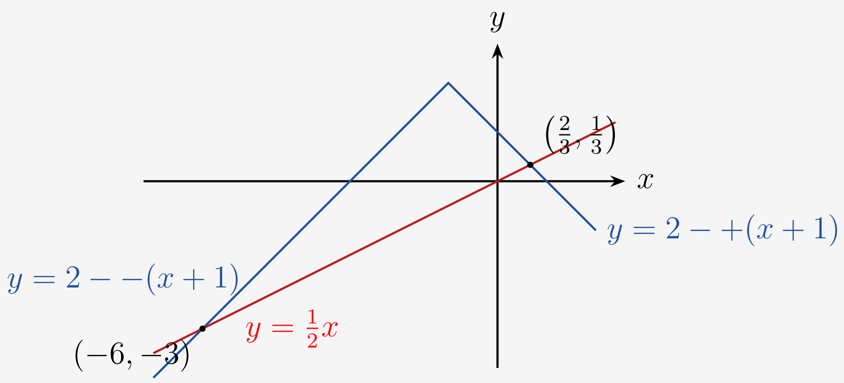
The point R is the positive intercept, so

$$R = (1, 0).$$

$$P = (-1, 2), \qquad Q = (0, 1), \qquad R = (1, 0).$$

(d) Solve $f(x) = \frac{1}{2}x$.

Use the graphs of $y = 2 - |x + 1|$ and $y = \frac{1}{2}x$. Their intersections give the solutions.



Now solve with each branch.

If $x > -1$, then $|x + 1| = x + 1$, so

$$2 - (x + 1) = \frac{1}{2}x$$

$$1 - x = \frac{1}{2}x$$

$$1 = \frac{3}{2}x$$

$$x = \frac{2}{3}.$$

If $x < -1$, then $|x+1| = -(x+1)$, so

$$2 - (-(x+1)) = \frac{1}{2}x$$

$$2 + x + 1 = \frac{1}{2}x$$

$$3 + x = \frac{1}{2}x$$

$$\frac{1}{2}x = -3$$

$$x = -6.$$

$$\boxed{x = -6 \quad \text{or} \quad x = \frac{2}{3}}$$

Question 4

Worked Solution

The function is

$$f(x) = |2x - 5|.$$

The function g is defined by

$$g(x) = x^2 - 4x + 1, \qquad 0 \le x \le 5.$$

(a) Sketch $y = f(x)$ and state where it cuts or meets the axes.

The vertex occurs when

$$2x - 5 = 0 \implies x = \frac{5}{2},$$

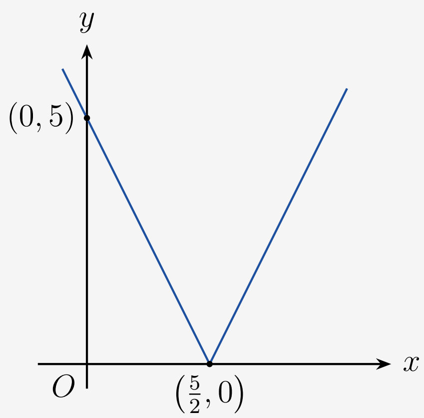
so the graph meets the x-axis at

$$\left(\frac{5}{2}, 0\right).$$

The y-intercept is

$$f(0) = |-5| = 5,$$

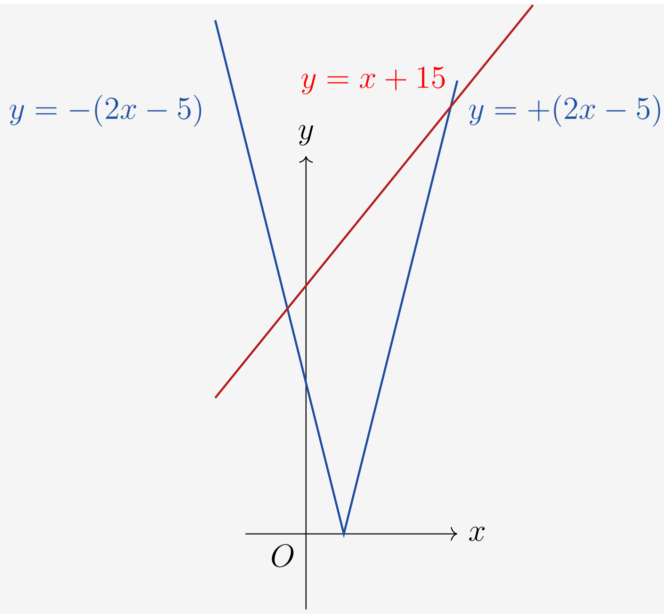
so the graph cuts the y-axis at $(0, 5)$.



The graph meets the x-axis at $\left(\frac{5}{2}, 0\right)$ and cuts the y-axis at $(0, 5)$.

(b) Solve $f(x) = 15 + x$.

Sketch $y = |2x - 5|$ and $y = x + 15$ on the same axes. Intersections give the solutions.



Solve using each branch.

For $x \geq \frac{5}{2}$, we have

$$2x - 5 = x + 15 \implies x = 20.$$

For $x < \frac{5}{2}$, we have

$$\begin{aligned} 5 - 2x &= x + 15 \\ -3x &= 10 \\ x &= -\frac{10}{3}. \end{aligned}$$

Both values satisfy the relevant branch conditions.

$$\boxed{x = -\frac{10}{3} \quad \text{or} \quad x = 20}$$

(c) Find $fg(2)$.

First find $g(2)$:

$$g(2) = 2^2 - 4(2) + 1 = 4 - 8 + 1 = -3.$$

Then apply f:

$$fg(2) = f(-3) = |2(-3) - 5| = |-11| = 11.$$

$$fg(2) = 11.$$

(d) Find the range of g.

Complete the square:

$$g(x) = x^2 - 4x + 1 = (x-2)^2 - 3.$$

Since $0 \le x \le 5$, the minimum value occurs at $x = 2$:

$$g_{\min} = -3.$$

At the endpoints,

$$g(0) = 1, \qquad g(5) = 25 - 20 + 1 = 6.$$

So the maximum value is 6.

$$\boxed{-3 \le g(x) \le 6}$$

Question 5

Worked Solution

The functions are

$$f(x) = \ln(2x - 1), \qquad x > \frac{1}{2},$$
$$g(x) = \frac{2}{x - 3}, \qquad x \neq 3.$$

(a) Find the exact value of $fg(4)$.

First

$$g(4) = \frac{2}{4 - 3} = 2.$$

Then

$$fg(4) = f(2) = \ln(2 \cdot 2 - 1) = \ln 3.$$

$$fg(4) = \ln 3.$$

(b) Find $f^{-1}(x)$, stating its domain.

Let

$$y = \ln(2x - 1).$$

Exponentiate both sides:

$$e^y = 2x - 1.$$

So

$$2x = e^y + 1$$
$$x = \frac{e^y + 1}{2}.$$

Therefore

$$f^{-1}(x) = \frac{e^x + 1}{2}.$$

The domain of f^{-1} is the range of f. Since $f(x) = \ln(2x - 1)$ can take any real value,

$$\text{domain of } f^{-1} = \mathbb{R}.$$

$$f^{-1}(x) = \frac{e^x + 1}{2}, \qquad \text{domain } \mathbb{R}.$$

(c) Sketch $y = |g(x)|$. State the vertical asymptote and the point where the graph crosses the y-axis.

Since

$$g(x) = \frac{2}{x - 3},$$

we have

$$|g(x)| = \left|\frac{2}{x-3}\right|.$$

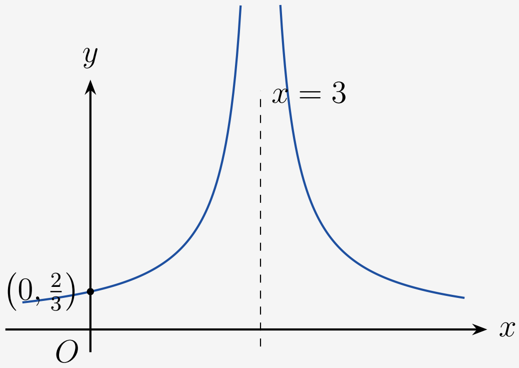
Taking the modulus reflects the part of the hyperbola below the x-axis above the x-axis. The vertical asymptote stays at

$$x = 3.$$

At the y-axis, $x = 0$:

$$y = \left|\frac{2}{0-3}\right| = \frac{2}{3}.$$

So the graph crosses the y-axis at $(0, \frac{2}{3})$.



Vertical asymptote: $x = 3$. y-intercept: $\left(0, \dfrac{2}{3}\right)$.

(d) Find the exact values of x for which $\left|\dfrac{2}{x-3}\right| = 3$.

Use the graphs $y = \left|\dfrac{2}{x-3}\right|$ and $y = 3$. The horizontal line $y = 3$ meets the curve in two points.

Solve algebraically:

$$\left|\frac{2}{x-3}\right| = 3 \implies \frac{2}{x-3} = 3 \quad \text{or} \quad \frac{2}{x-3} = -3.$$

From

$$\frac{2}{x-3} = 3,$$

we get

$$2 = 3(x-3) \implies 3x = 11 \implies x = \frac{11}{3}.$$

From
$$\frac{2}{x-3} = -3,$$
we get
$$2 = -3(x-3) \implies -3x = -7 \implies x = \frac{7}{3}.$$

$$\boxed{x = \frac{7}{3} \quad \text{or} \quad x = \frac{11}{3}}$$

Question 6

Worked Solution

$$f(x) = 2|3 - x| + 5, \qquad x \geq 0.$$

(a) State the range of f.

The smallest value of $|3 - x|$ is 0, which occurs when $x = 3$. So the minimum value of $f(x)$ is

$$2(0) + 5 = 5.$$

Since the graph rises on both sides of the vertex and the domain is unbounded to the right,

$$f(x) \geq 5.$$

$$\boxed{f(x) \geq 5}$$

(b) Solve $f(x) = \frac{1}{2}x + 30$.

Use the graphs of $y = 2|3 - x| + 5$ and $y = \frac{1}{2}x + 30$.

For $x \geq 3$, the modulus graph has equation

$$y = 2(x - 3) + 5 = 2x - 1.$$

For $0 \leq x < 3$, it has equation

$$y = 2(3 - x) + 5 = 11 - 2x.$$

Solve on each branch.

Right-hand branch:

$$2x - 1 = \frac{1}{2}x + 30$$
$$\frac{3}{2}x = 31$$
$$x = \frac{62}{3}.$$

Left-hand branch:

$$11 - 2x = \frac{1}{2}x + 30$$
$$-\frac{5}{2}x = 19$$
$$x = -\frac{38}{5},$$

which is not allowed since the domain is $x \geq 0$.

$$\boxed{x = \frac{62}{3}}$$

(c) Given that $f(x) = k$ has two distinct roots, state the possible values of k.

A horizontal line $y = k$ must cut the graph in two distinct places.

The vertex is at $(3, 5)$, so we need

$$k > 5$$

otherwise the line touches at only one point or misses the graph.

Because the domain starts at $x = 0$, the left-hand branch only runs up to the point

$$f(0) = 2|3| + 5 = 11.$$

If $k > 11$, the horizontal line lies above the left endpoint and meets only the right-hand branch once.

Therefore we need

$$5 < k \leq 11.$$

$$\boxed{5 < k \leq 11}$$

Question 7

Worked Solution

The functions are

$$f(x) = 2|x| + 3, \qquad g(x) = 3 - 4x.$$

(a) State the range of f.

Since $|x| \geq 0$, we have

$$2|x| + 3 \geq 3.$$

The minimum value 3 occurs when $x = 0$.

$$\boxed{f(x) \geq 3}$$

(b) Find $fg(1)$.

First

$$g(1) = 3 - 4(1) = -1.$$

Then

$$fg(1) = f(-1) = 2|-1| + 3 = 2 + 3 = 5.$$

$$fg(1) = 5.$$

(c) Find g^{-1}, the inverse function of g.

Let

$$y = 3 - 4x.$$

Then

$$4x = 3 - y \implies x = \frac{3-y}{4}.$$

Hence

$$g^{-1}(x) = \frac{3-x}{4}.$$

$$g^{-1}(x) = \frac{3-x}{4}.$$

(d) Solve $gg(x) + [g(x)]^2 = 0$.

First find each part:

$$g(x) = 3 - 4x,$$
$$gg(x) = g(3-4x) = 3 - 4(3-4x) = 16x - 9,$$
$$[g(x)]^2 = (3-4x)^2 = 9 - 24x + 16x^2.$$

So

$$gg(x) + [g(x)]^2 = 0$$

becomes

$$16x - 9 + 9 - 24x + 16x^2 = 0.$$

Hence

$$16x^2 - 8x = 0$$

$$8x(2x - 1) = 0.$$

So

$$x = 0 \qquad \text{or} \qquad x = \frac{1}{2}.$$

$$\boxed{x = 0 \;\text{ or }\; x = \frac{1}{2}}$$

Question 8

Worked Solution

Here a and b are positive constants.

(a)(i) Sketch $y = |2x - a|$.

The vertex occurs when $2x - a = 0$, so

$$x = \frac{a}{2}.$$

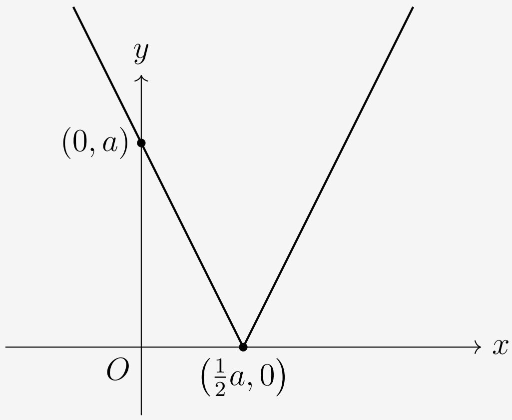
Hence the graph has vertex

$$\left(\frac{a}{2}, 0\right).$$

At the y-axis,

$$y = |2(0) - a| = a,$$

so the graph crosses the y-axis at $(0, a)$.



(a)(ii) Sketch $y = |2x - a| + b$.

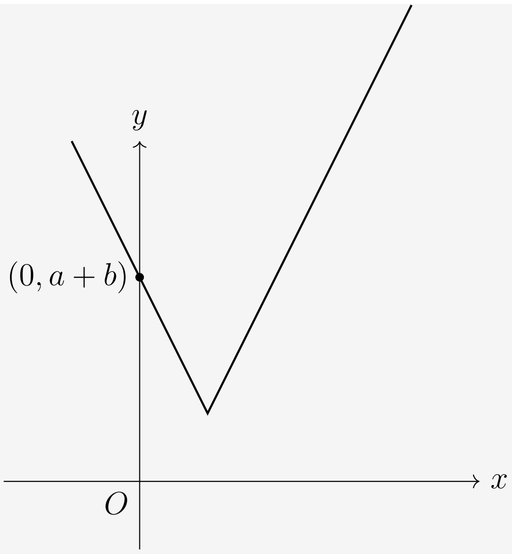
This is the graph from part (i) shifted up by b units. So the vertex is

$$\left(\frac{a}{2}, b\right)$$

and the y-intercept is

$$(0, a + b).$$

There is no x-intercept because $b > 0$.

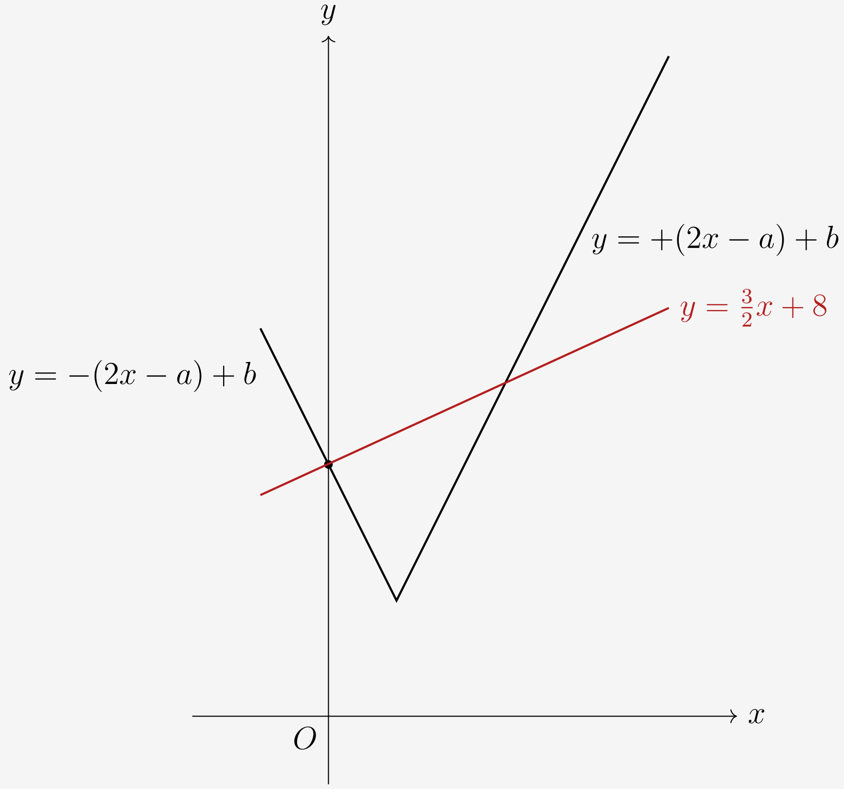


For $y = |2x - a|$, the intercepts are $\left(\frac{a}{2}, 0\right)$ and $(0, a)$.
For $y = |2x - a| + b$, the graph is translated up, so the key points are $\left(\frac{a}{2}, b\right)$ and $(0, a + b)$.

(b) Given that $|2x - a| + b = \frac{3}{2}x + 8$ has a solution at $x = 0$ and a solution at $x = c$, find c in terms of a.

Use the graphs of

$$y = |2x - a| + b \qquad \text{and} \qquad y = \frac{3}{2}x + 8.$$



Since $x = 0$ is a solution,

$$|2(0) - a| + b = \frac{3}{2}(0) + 8.$$

Because $a > 0$, $|-a| = a$, so

$$a + b = 8.$$

Now find the other intersection using the right-hand branch of the modulus graph:

$$2x - a + b = \frac{3}{2}x + 8.$$

At the second intersection $x = c$, this gives

$$2c - a + b = \frac{3}{2}c + 8.$$

Using $a + b = 8$,

$$2c - a + (8 - a) = \frac{3}{2}c + 8$$

$$2c - 2a = \frac{3}{2}c$$

$$\frac{1}{2}c = 2a$$

$$c = 4a.$$

$$\boxed{c = 4a}$$

Question 9

Worked Solution

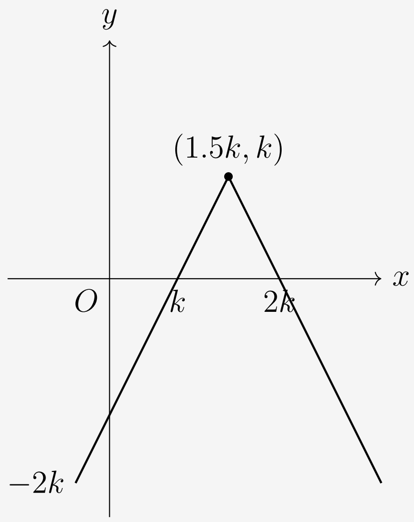
The graph shown is
$$y = |2x - 3k|, \qquad k > 0.$$
Define
$$f(x) = k - |2x - 3k|.$$

(a) Sketch $y = f(x)$, stating the maximum point and any intercepts.

Subtracting the modulus graph from k reflects it in the x-axis and shifts it up by k. So $y = f(x)$ is an inverted V-shape.

The maximum occurs when the modulus part is zero:
$$2x - 3k = 0 \implies x = \frac{3k}{2}.$$
Then
$$y = k.$$
So the maximum point is
$$\left(\frac{3k}{2}, k\right).$$
For the x-intercepts:
$$k - |2x - 3k| = 0 \implies |2x - 3k| = k.$$
So
$$2x - 3k = k \quad \text{or} \quad 2x - 3k = -k.$$
This gives
$$x = 2k \quad \text{or} \quad x = k.$$
Hence the graph cuts the x-axis at $(k, 0)$ and $(2k, 0)$.

For the y-intercept, put $x = 0$:
$$y = k - |-3k| = k - 3k = -2k.$$
So the graph cuts the y-axis at $(0, -2k)$.



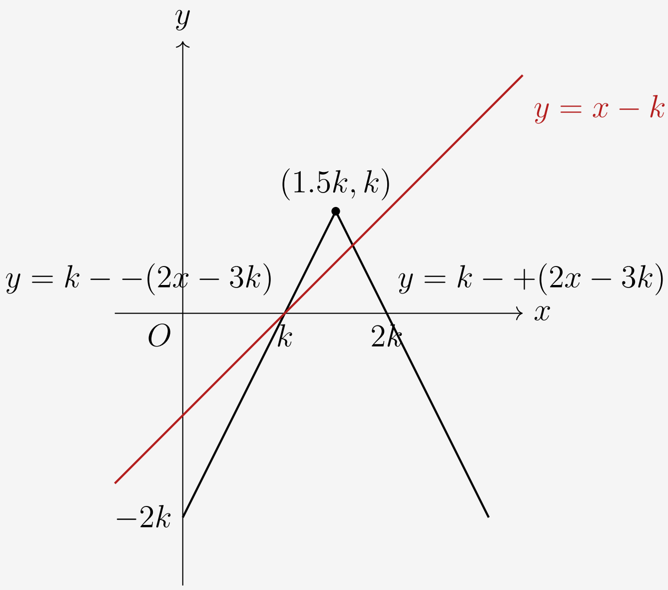
Maximum point: $\left(\frac{3k}{2}, k\right)$.
Intercepts: $(k, 0)$, $(2k, 0)$ and $(0, -2k)$.

(b) Find, in terms of k, the set of values of x for which $k - |2x - 3k| > x - k$.

Use the graphs of

$$y = k - |2x - 3k| \qquad \text{and} \qquad y = x - k.$$

We want where the inverted V lies above the line.



If $x \geq \frac{3k}{2}$, then $|2x - 3k| = 2x - 3k$, so

$$k - (2x - 3k) > x - k$$

$$4k - 2x > x - k$$

$$5k > 3x$$
$$x < \frac{5k}{3}.$$

Together with $x \geq \frac{3k}{2}$, this gives

$$\frac{3k}{2} \leq x < \frac{5k}{3}.$$

If $x < \frac{3k}{2}$, then $|2x - 3k| = 3k - 2x$, so

$$k - (3k - 2x) > x - k$$
$$2x - 2k > x - k$$
$$x > k.$$

Together with $x < \frac{3k}{2}$, this gives

$$k < x < \frac{3k}{2}.$$

Combining both intervals,

$$k < x < \frac{5k}{3}.$$

$$\boxed{\{x \in \mathbb{R} : k < x < \tfrac{5k}{3}\}}$$

(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation $y = 3 - 5f\left(\frac{1}{2}x\right)$.

First substitute $\frac{1}{2}x$ into f:

$$f\left(\frac{1}{2}x\right) = k - \left|2\left(\frac{1}{2}x\right) - 3k\right| = k - |x - 3k|.$$

Therefore

$$y = 3 - 5\left(k - |x - 3k|\right) = 3 - 5k + 5|x - 3k|.$$

This is an upright V-shape. Its minimum occurs when the modulus part is zero:

$$x - 3k = 0 \implies x = 3k.$$

Then

$$y = 3 - 5k.$$

$$\boxed{(3k,\, 3 - 5k)}$$

End of Worked Solutions