Question 1

(Jun 2005, Q5)

Worked Solution

Solve $2\cos 2x = 1 + \cos x$ for $0° \leq x < 360°$.

Use the double angle formula $\cos 2x = 2\cos^2 x - 1$:

$$2(2\cos^2 x - 1) = 1 + \cos x$$

$$4\cos^2 x - 2 = 1 + \cos x$$

$$4\cos^2 x - \cos x - 3 = 0$$

$$(4\cos x + 3)(\cos x - 1) = 0$$

$$\cos x = -\tfrac{3}{4} \quad \text{or} \quad \cos x = 1$$

For $\cos x = -\frac{3}{4}$: $x = 138.6°$ or $x = 221.4°$

For $\cos x = 1$: $x = 0°$

$x = 0°,\ 138.6°,\ 221.4°$

Question 2

(Jan 2006, Q4)

Worked Solution

Solve $2\sin 2\theta + \cos 2\theta = 1$ for $0° \leq \theta < 360°$.

Use $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$:

$$4\sin\theta\cos\theta + 1 - 2\sin^2\theta = 1$$

$$4\sin\theta\cos\theta - 2\sin^2\theta = 0$$

$$2\sin\theta(2\cos\theta - \sin\theta) = 0$$

Case 1: $\sin\theta = 0 \Rightarrow \theta = 0°,\ 180°$ (Not 360 as this excluded in the inequality in the first line.)

Case 2: $2\cos\theta - \sin\theta = 0 \Rightarrow 2\cos\theta = \sin\theta \ \Rightarrow 2 = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = 2$

$$\theta = 63.4° \quad \text{or} \quad \theta = 243.4°$$

$\theta = 0°,\ 63.4°,\ 180°,\ 243.4°$

Question 3

(Jun 2006, Q3)

Worked Solution

Given that $\sin(\theta + \alpha) = 2\sin\theta$, show that $\tan\theta = \dfrac{\sin\alpha}{2 - \cos\alpha}$.

Hence solve $\sin(\theta + 40°) = 2\sin\theta$ for $0° \leq \theta \leq 360°$.

Expand the left side using the compound angle formula:

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = 2\sin\theta$$

Divide through by $\cos\theta$:

$$\tan\theta\cos\alpha + \sin\alpha = 2\tan\theta$$

$$\sin\alpha = 2\tan\theta - \tan\theta\cos\alpha = \tan\theta(2 - \cos\alpha)$$

$$\tan\theta = \frac{\sin\alpha}{2 - \cos\alpha} \qquad \checkmark$$

With $\alpha = 40°$:

$$\tan\theta = \frac{\sin 40°}{2 - \cos 40°} = \frac{0.6428}{2 - 0.7660} = \frac{0.6428}{1.2340} \approx 0.5209$$

$$\theta = 27.5° \quad \text{or} \quad \theta = 207.5°$$

$\theta = 27.5°,\ 207.5°$

Question 4

(Jan 2007, Q3)

Worked Solution

(i) Use the formula for $\sin(\theta+\phi)$ with $\theta = 45°$, $\phi = 60°$ to show that $\sin 105° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

$\sin 60° = \dfrac{\sqrt{3}}{2}, \quad \cos 60° = \dfrac{1}{2}, \quad \sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$

$$\sin 105° = \sin(60° + 45°) = \sin 60° \cos 45° + \cos 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad \checkmark$$

(ii) In triangle ABC, angle $BAC = 45°$, angle $ACB = 30°$, $AB = 1$. Show $AC = \dfrac{\sqrt{3}+1}{\sqrt{2}}$.

Angle $ABC = 180° - 45° - 30° = 105°$. By the sine rule:

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \implies \frac{AC}{\sin 105°} = \frac{1}{\sin 30°}$$

$$AC = \frac{\sin 105°}{\sin 30°} = \frac{\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{1}{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times 2 = \frac{\sqrt{3}+1}{\sqrt{2}} \qquad \checkmark$$

Question 5

(Jan 2008, Q4)

Worked Solution

The angle θ satisfies $\sin(\theta + 45°) = \cos\theta$.

(i) Using exact values of $\sin 45°$ and $\cos 45°$, show that $\tan\theta = \sqrt{2} - 1$.

$$\sin\theta\cos 45° + \cos\theta\sin 45° = \cos\theta$$

$$\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \cos\theta$$

$$\frac{1}{\sqrt{2}}\sin\theta = \cos\theta - \frac{1}{\sqrt{2}}\cos\theta = \cos\theta\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2} - 1 \qquad \checkmark$$

(ii) Find the values of θ for $0° < \theta < 360°$.

$$\tan\theta = \sqrt{2} - 1 \approx 0.4142$$

$$\theta = 22.5° \quad \text{or} \quad \theta = 202.5°$$

$\theta = 22.5°,\ 202.5°$

Question 6

(Jun 2012, Q5)

Worked Solution

Given $\sin(x + 45°) = 2\cos x$, show that $\sin x + \cos x = 2\sqrt{2}\cos x$, and hence solve for $0° \leq x \leq 360°$.

$$\sin x \cos 45° + \cos x \sin 45° = 2\cos x$$
$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 2\cos x$$

Multiply through by $\sqrt{2}$:

$$\sin x + \cos x = 2\sqrt{2}\cos x \qquad \checkmark$$

Rearranging:

$$\sin x = (2\sqrt{2} - 1)\cos x$$
$$\tan x = 2\sqrt{2} - 1 \approx 1.828$$
$$x = 61.32° \quad \text{or} \quad x = 241.32°$$

$x = 61.32°, \; 241.32°$

Question 7

(Jun 2013, Q3)

Worked Solution

Using appropriate right-angled triangles, show $\tan 45° = 1$ and $\tan 30° = \frac{1}{\sqrt{3}}$, then show $\tan 75° = 2 + \sqrt{3}$.

From a 45-45-90 triangle (sides 1, 1, $\sqrt{2}$): $\tan 45° = \frac{1}{1} = 1$ ✓

From a 30-60-90 triangle (sides 1, $\sqrt{3}$, 2): $\tan 30° = \frac{1}{\sqrt{3}}$ ✓

Using the compound angle formula for $\tan(45° + 30°)$:

$$\tan 75° = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Rationalise by multiplying numerator and denominator by $(\sqrt{3} + 1)$:

$$= \frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2 - 1^2} = \frac{3 + 2\sqrt{3} + 1}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} \qquad ✓$$

Question 8

(Jun 2015, Q2)

Worked Solution

Express $6\cos 2\theta + \sin\theta$ in terms of $\sin\theta$. Hence solve $6\cos 2\theta + \sin\theta = 0$ for $0° \leq \theta \leq 360°$.

Use $\cos 2\theta = 1 - 2\sin^2\theta$:

$$6(1 - 2\sin^2\theta) + \sin\theta = 6 - 12\sin^2\theta + \sin\theta$$

Setting equal to zero:

$$12\sin^2\theta - \sin\theta - 6 = 0$$

$$(4\sin\theta - 3)(3\sin\theta + 2) = 0$$

$$\sin\theta = \tfrac{3}{4} \quad \text{or} \quad \sin\theta = -\tfrac{2}{3}$$

For $\sin\theta = \frac{3}{4}$: $\theta = 48.6°$ or $\theta = 131.4°$

For $\sin\theta = -\frac{2}{3}$: $\theta = 221.8°$ or $\theta = 318.2°$

$\theta = 48.6°,\ 131.4°,\ 221.8°,\ 318.2°$

Question 9

(Jun 2016, Q4)

Worked Solution

Solve $2\sin 2\theta = 1 + \cos 2\theta$ for $0° \le \theta \le 180°$.

Use $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$:

$$4\sin\theta\cos\theta = 1 + 2\cos^2\theta - 1$$

$$4\sin\theta\cos\theta = 2\cos^2\theta$$

$$2\cos\theta(2\sin\theta - \cos\theta) = 0$$

Case 1: $\cos\theta = 0 \Rightarrow \theta = 90°$

Case 2: $2\sin\theta = \cos\theta \Rightarrow \tan\theta = \dfrac{1}{2}$

$$\theta = 26.6°$$

$\theta = 26.6°,\ 90°$

End of Worked Solutions