Question 1

(OCR H240/03, Sample Q4)

Worked Solution

Show that, for small θ (in radians):

$$1 + \cos\theta - 3\cos^2\theta \approx -1 + \tfrac{5}{2}\theta^2$$

Small angle approximations: $\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \dfrac{\theta^2}{2}, \quad \tan\theta \approx \theta$:

$$1 + \cos\theta - 3\cos^2\theta \approx 1 + \left(1 - \frac{\theta^2}{2}\right) - 3\left(1 - \frac{\theta^2}{2}\right)^2$$

$$= 1 + 1 - \frac{\theta^2}{2} - 3\left(1 - \theta^2 + \frac{\theta^4}{4}\right)$$

$$= 2 - \frac{\theta^2}{2} - 3 + 3\theta^2 - \frac{3\theta^4}{4}$$

Since θ is small, neglect the θ^4 term:

$$\approx -1 + \frac{5}{2}\theta^2 \qquad \checkmark$$

Question 2

(OCR H240/03, Practice Paper Set 1, Q3)

Worked Solution

Triangle ABC has angle $A = \theta$ radians, angle $B = \frac{3}{4}\pi$ radians, $AB = 1$ unit.

(i) Use the sine rule to show that $AC = \dfrac{1}{\cos\theta - \sin\theta}$.

Angle $C = \pi - \theta - \frac{3}{4}\pi = \frac{1}{4}\pi - \theta$

By the sine rule:

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \implies AC = \frac{\sin\frac{3}{4}\pi}{\sin(\frac{1}{4}\pi - \theta)}$$

Expand using compound angle formula:

$$\sin\left(\frac{\pi}{4} - \theta\right) = \sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$$

Also $\sin\frac{3}{4}\pi = \dfrac{1}{\sqrt{2}}$, so:

$$AC = \frac{1/\sqrt{2}}{(1/\sqrt{2})(\cos\theta - \sin\theta)} = \frac{1}{\cos\theta - \sin\theta} \qquad \checkmark$$

(ii) Given θ is small, use part (i) to show $AC \approx 1 + p\theta + q\theta^2$ and find p and q.

Use $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$ and $\sin\theta \approx \theta$:

$$\cos\theta - \sin\theta \approx 1 - \frac{\theta^2}{2} - \theta = 1 - \theta - \frac{\theta^2}{2}$$

$$AC = \frac{1}{1 - \theta - \frac{\theta^2}{2}} = \left(1 - \theta - \frac{\theta^2}{2}\right)^{-1}$$

Binomial expansion (keeping terms up to θ^2):

$$\approx 1 + \left(\theta + \frac{\theta^2}{2}\right) + \left(\theta + \frac{\theta^2}{2}\right)^2 + \cdots \approx 1 + \theta + \frac{\theta^2}{2} + \theta^2 = 1 + \theta + \frac{3}{2}\theta^2$$

$$p = 1, \quad q = \frac{3}{2}$$

Question 3

(OCR H240/02, Practice Paper Set 3, Q3)

Worked Solution

Use small angle approximations to estimate the solution of $\dfrac{\cos\frac{1}{2}\theta}{1+\sin\theta} = 0.825$, neglecting θ^3 and above.

Small angle approximations: $\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \dfrac{\theta^2}{2}, \quad \tan\theta \approx \theta$. Use $\cos\frac{1}{2}\theta \approx 1 - \dfrac{(\theta/2)^2}{2} = 1 - \dfrac{\theta^2}{8}$, $\sin\theta \approx \theta$:

$$\frac{1 - \frac{\theta^2}{8}}{1+\theta} = 0.825$$

$$1 - \frac{\theta^2}{8} = 0.825(1+\theta) = 0.825 + 0.825\theta$$

$$0.125\theta^2 + 0.825\theta - 0.175 = 0$$

Multiplying by 8: $\theta^2 + 6.6\theta - 1.4 = 0$... or equivalently $0.125\theta^2 + 0.825\theta - 0.175 = 0$. Using the quadratic formula:

$$\theta = \frac{-0.825 \pm \sqrt{0.825^2 + 4(0.125)(0.175)}}{2(0.125)} = \frac{-0.825 \pm \sqrt{0.6806 + 0.0875}}{0.25}$$

$$= \frac{-0.825 \pm 0.877}{0.25}$$

Taking the positive root: $\theta = \dfrac{0.052}{0.25} \approx 0.206$

Discard $\theta \approx -6.81$ as not small.

$\theta \approx 0.206$

Question 4

(Edexcel Practice Paper 5, Q2)

Worked Solution

(a) When θ is small, show that $\dfrac{1+\sin\theta+\tan 2\theta}{2\cos 3\theta - 1}$ can be written as $\dfrac{1}{1-3\theta}$.

Small angle approximations: $\sin\theta \approx \theta, \quad \cos\theta \approx 1-\dfrac{\theta^2}{2}, \quad \tan\theta \approx \theta$ with small θ:

$$1+\sin\theta+\tan 2\theta \approx 1+\theta+2\theta = 1+3\theta$$

$$2\cos 3\theta - 1 \approx 2\left(1-\frac{(3\theta)^2}{2}\right) - 1 = 2-9\theta^2-1 = 1-9\theta^2$$
$$= (1-3\theta)(1+3\theta)$$

Therefore:

$$\frac{1+3\theta}{(1-3\theta)(1+3\theta)} = \frac{1}{1-3\theta} \qquad \checkmark$$

(b) Hence write down the value of $\dfrac{1+\sin\theta+\tan 2\theta}{2\cos 3\theta - 1}$ when θ is small.

As $\theta \to 0$: $\dfrac{1}{1-3\theta} \to \dfrac{1}{1-0} = 1$

The value approaches 1 as $\theta \to 0$.

Question 5

(Edexcel A Level Maths, Jun 2018, Paper 1, Q1)

Worked Solution

Given θ is small and in radians, find an approximate value of:

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

Small angle approximations: $\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}, \quad \tan\theta \approx \theta$:

$$\cos 4\theta \approx 1 - \frac{(4\theta)^2}{2} = 1 - 8\theta^2 \implies 1 - \cos 4\theta \approx 8\theta^2$$

$$\sin 3\theta \approx 3\theta$$

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta} \approx \frac{8\theta^2}{2\theta \cdot 3\theta} = \frac{8\theta^2}{6\theta^2} = \frac{4}{3}$$

$$\frac{4}{3}$$

Question 6

(OCR H640/03, Jun 2018, Q3)

Worked Solution

A circle has centre O and radius 1 unit. Points A and B lie on the circle with angle $AOB = \theta$ radians. C lies on AO, and BC is perpendicular to AO.

Show that, when θ is small, $AC \approx \frac{1}{2}\theta^2$.

Since $OA = OB = 1$ and angle $AOB = \theta$, point C is the foot of the perpendicular from B to OA.

$OC = OB\cos\theta = \cos\theta$

Therefore: $AC = AO - OC = 1 - \cos\theta$

Using the small angle approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$:

$$AC = 1 - \cos\theta \approx 1 - \left(1 - \frac{\theta^2}{2}\right) = \frac{\theta^2}{2} = \frac{1}{2}\theta^2 \qquad \checkmark$$

Question 7

(Edexcel Mock Paper 2, Q1)

Worked Solution

(a) Given θ is small and in radians, show that

$$\cos\theta - \sin\left(\tfrac{1}{2}\theta\right) + 2\tan\theta = \tfrac{11}{10} \qquad \text{(I)}$$

can be written as $5\theta^2 - 15\theta + 1 \approx 0$.

Small angle approximations: $\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \dfrac{\theta^2}{2}, \quad \tan\theta \approx \theta$:

$$\left(1 - \frac{\theta^2}{2}\right) - \frac{\theta}{2} + 2\theta \approx \frac{11}{10}$$

$$1 - \frac{\theta^2}{2} + \frac{3\theta}{2} \approx \frac{11}{10}$$

$$-\frac{\theta^2}{2} + \frac{3\theta}{2} - \frac{1}{10} \approx 0$$

Multiply through by -10:

$$5\theta^2 - 15\theta + 1 \approx 0 \qquad \checkmark$$

(b) The solutions of $5\theta^2 - 15\theta + 1 = 0$ are $\theta = 0.068$ and $\theta = 2.932$ (to 3 d.p.). Comment on the validity of each as an approximate solution to equation (I).

$\theta = 0.068$: this value is small (close to 0), so the small angle approximations are valid. This is a valid approximate solution.

$\theta = 2.932$: this value is large, so the small angle approximations are not valid for this value. This is **not** a valid approximate solution to equation (I).

End of Worked Solutions