Question 1

(Jun 2005, Q7 – OCR 4723)

Worked Solution

(i) Write down $\cos 2x$ in terms of $\cos x$: $\cos 2x = 2\cos^2 x - 1$

(ii) Prove that $\dfrac{4\cos 2x}{1+\cos 2x} \equiv 4 - 2\sec^2 x$.

$$\frac{4\cos 2x}{1+\cos 2x} = \frac{4(2\cos^2 x - 1)}{1 + (2\cos^2 x - 1)} = \frac{8\cos^2 x - 4}{2\cos^2 x}$$
$$= 4 - \frac{4}{2\cos^2 x} \cdot \frac{1}{1} = 4 - \frac{2}{\cos^2 x} = 4 - 2\sec^2 x \qquad \checkmark$$

(iii) Solve $\dfrac{4\cos 2x}{1+\cos 2x} = 3\tan x - 7$ for $0 < x < 2\pi$.

Using the result from part (ii):

$$4 - 2\sec^2 x = 3\tan x - 7$$

Use $\sec^2 x = 1 + \tan^2 x$:

$$4 - 2(1 + \tan^2 x) = 3\tan x - 7$$
$$2 - 2\tan^2 x = 3\tan x - 7$$
$$2\tan^2 x + 3\tan x - 9 = 0$$
$$(2\tan x - 3)(\tan x + 3) = 0$$
$$\tan x = \tfrac{3}{2} \quad \text{or} \quad \tan x = -3$$

For $\tan x = \frac{3}{2}$: $x = 0.983,\ 4.12$

For $\tan x = -3$: $x = 1.89,\ 5.03$

$x \approx 0.983,\ 1.89,\ 4.12,\ 5.03$

Question 2

(Jun 2006, Q5 – OCR 4723)

Worked Solution

(i) Write down $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$: $\sin 2\theta = 2\sin\theta\cos\theta$

(ii) Given $\sin\alpha = \frac{1}{4}$ and α is acute, show $\sin 2\alpha = \frac{1}{8}\sqrt{15}$.

Since $\sin\alpha = \frac{1}{4}$ and α acute: $\cos\alpha = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \cdot \frac{1}{4} \cdot \frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{8} = \tfrac{1}{8}\sqrt{15} \qquad \checkmark$$

(iii) Solve $5\sin 2\beta \sec\beta = 3$ for $0° < \beta < 90°$.

$$5 \cdot 2\sin\beta\cos\beta \cdot \frac{1}{\cos\beta} = 3$$

$$10\sin\beta = 3 \implies \sin\beta = 0.3$$

$$\beta = 17.5°$$

$\beta = 17.5°$

Question 3

(Jan 2007, Q2 – OCR 4723)

Worked Solution

Given $\sin\theta = \dfrac{12}{13}$ and θ is acute, find exact values of:

(i) $\cot\theta$

Since $\sin\theta = \frac{12}{13}$: $\cos\theta = \frac{5}{13}$ (by Pythagoras, $5^2 + 12^2 = 13^2$)

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{5/13}{12/13} = \frac{5}{12}$$

$\cot\theta = \dfrac{5}{12}$

(ii) $\cos 2\theta$

Using $\cos 2\theta = 1 - 2\sin^2\theta$:

$$\cos 2\theta = 1 - 2\left(\frac{12}{13}\right)^2 = 1 - 2\cdot\frac{144}{169} = 1 - \frac{288}{169} = -\frac{119}{169}$$

$\cos 2\theta = -\dfrac{119}{169}$

Question 4

(Jun 2008, Q5 – OCR 4723)

Worked Solution

(a) Express $\tan 2\alpha$ in terms of $\tan\alpha$ and solve $\tan 2\alpha \tan\alpha = 8$ for $0° < \alpha < 180°$.

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\tan\alpha = 8$$

$$\frac{2\tan^2\alpha}{1-\tan^2\alpha} = 8$$

$$2\tan^2\alpha = 8(1-\tan^2\alpha) = 8 - 8\tan^2\alpha$$

$$10\tan^2\alpha = 8 \implies \tan^2\alpha = \tfrac{4}{5} \implies \tan\alpha = \pm\sqrt{\tfrac{4}{5}}$$

For $0° < \alpha < 180°$: $\tan\alpha = \sqrt{\tfrac{4}{5}}$ gives $\alpha = 41.8°$; $\tan\alpha = -\sqrt{\tfrac{4}{5}}$ gives $\alpha = 138.2°$

$\alpha = 41.8°,\ 138.2°$

(b) Given $\sin\beta = \tfrac{6}{7}$ and β acute:

(i) $\beta = \dfrac{1}{\sin\beta} = \dfrac{7}{6}$

(ii) $\cot^2\beta =^2 \beta - 1 = \dfrac{49}{36} - 1 = \dfrac{13}{36}$

$\beta = \dfrac{7}{6};\quad \cot^2\beta = \dfrac{13}{36}$

Question 5

(Jan 2009, Q3 – OCR 4723)

Worked Solution

(i) Express $2\tan^2\theta - \dfrac{1}{\cos\theta}$ in terms of $\sec\theta$.

Use $\tan^2\theta = \sec^2\theta - 1$ and $\dfrac{1}{\cos\theta} = \sec\theta$:

$$2(\sec^2\theta - 1) - \sec\theta = 2\sec^2\theta - \sec\theta - 2$$

$2\sec^2\theta - \sec\theta - 2$

(ii) Solve $2\tan^2\theta - \dfrac{1}{\cos\theta} = 4$ for $0° < \theta < 360°$.

$$2\sec^2\theta - \sec\theta - 2 = 4$$
$$2\sec^2\theta - \sec\theta - 6 = 0$$
$$(2\sec\theta + 3)(\sec\theta - 2) = 0$$
$$\sec\theta = -\tfrac{3}{2} \implies \cos\theta = -\tfrac{2}{3} : \quad \theta = 131.8°,\ 228.2°$$
$$\sec\theta = 2 \implies \cos\theta = \tfrac{1}{2} : \quad \theta = 60°,\ 300°$$

$\theta = 60°,\ 131.8°,\ 228.2°,\ 300°$

Question 6

(Jan 2010, Q2 – OCR 4723)

Worked Solution

$0° < \theta < 90°$.

(i) Given $6\sin 2\theta = 5\cos\theta$, find the exact value of $\sin\theta$.

$$6 \cdot 2\sin\theta\cos\theta = 5\cos\theta$$

$$12\sin\theta\cos\theta = 5\cos\theta$$

Since $\cos\theta \neq 0$:

$$\sin\theta = \frac{5}{12}$$

$$\sin\theta = \frac{5}{12}$$

(ii) Given $8\cos\theta^2\theta = 3$, find the exact value of $\cos\theta$.

Use $^2\theta = 1 + \cot^2\theta = \frac{1}{\sin^2\theta}$, and $\sin^2\theta = 1 - \cos^2\theta$:

$$8\cos\theta \cdot \frac{1}{1-\cos^2\theta} = 3$$

$$8\cos\theta = 3(1-\cos^2\theta) = 3 - 3\cos^2\theta$$

$$3\cos^2\theta + 8\cos\theta - 3 = 0$$

$$(3\cos\theta - 1)(\cos\theta + 3) = 0$$

Since $0° < \theta < 90°$, $\cos\theta > 0$, so $\cos\theta = \frac{1}{3}$

$$\cos\theta = \frac{1}{3}$$

Question 7

(Jun 2010, Q3 – OCR 4723)

Worked Solution

(i) Express $\theta(3\cos 2\theta + 7) + 11 = 0$ in the form $a\sin^2\theta + b\sin\theta + c = 0$.

Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\theta = \dfrac{1}{\sin\theta}$:

$$\frac{3(1 - 2\sin^2\theta) + 7}{\sin\theta} + 11 = 0$$

$$\frac{10 - 6\sin^2\theta}{\sin\theta} + 11 = 0$$

Multiply through by $\sin\theta$:

$$10 - 6\sin^2\theta + 11\sin\theta = 0$$

$$6\sin^2\theta - 11\sin\theta - 10 = 0$$

(ii) Solve for $-180° < \theta < 180°$:

$$(6\sin\theta + \cdot)(\sin\theta - \cdot) \to \text{ using quadratic formula or factorisation}$$

$$(2\sin\theta - 5)(3\sin\theta + 2) = 0 \quad \Rightarrow \quad \sin\theta = \frac{5}{2} \text{ (reject)}, \quad \sin\theta = -\frac{2}{3}$$

$$\theta = -41.8° \quad \text{or} \quad \theta = -138.2°$$

$\theta = -41.8°,\ -138.2°$

Question 8

(Jan 2012, Q4 – OCR 4723)

Worked Solution

Given $2\cot\alpha = 1$ and $24 + \sec^2\beta = 10\tan\beta$, with α, β acute:

(i) Find $\tan\alpha$ and $\tan\beta$.

$2\cot\alpha = 1 \Rightarrow \cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$

For $\tan\beta$: use $\sec^2\beta = 1 + \tan^2\beta$:

$$24 + 1 + \tan^2\beta = 10\tan\beta$$

$$\tan^2\beta - 10\tan\beta + 25 = 0 \implies (\tan\beta - 5)^2 = 0 \implies \tan\beta = 5$$

$\tan\alpha = 2; \quad \tan\beta = 5$

(ii) Find the exact value of $\tan(\alpha + \beta)$.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{2+5}{1-2\times 5} = \frac{7}{1-10} = \frac{7}{-9} = -\frac{7}{9}$$

$\tan(\alpha+\beta) = -\dfrac{7}{9}$

Question 9

(Jan 2013, Q2 – OCR 4723)

Worked Solution

The acute angle A is such that $\tan A = 2$.

(i) Find the exact value of A.

From $\tan A = 2$: opposite $= 2$, adjacent $= 1$, hypotenuse $= \sqrt{5}$

$$\sin A = \frac{2}{\sqrt{5}} \implies A = \frac{\sqrt{5}}{2}$$

$$A = \frac{\sqrt{5}}{2}$$

(ii) The angle B is such that $\tan(A + B) = 3$. Find the exact value of $\tan B$.

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 3 \implies \frac{2 + \tan B}{1 - 2\tan B} = 3$$

$$2 + \tan B = 3(1 - 2\tan B) = 3 - 6\tan B$$

$$7\tan B = 1 \implies \tan B = \frac{1}{7}$$

$$\tan B = \frac{1}{7}$$

Question 10

(Jun 2013, Q2 – OCR 4723)

Worked Solution

Using appropriate identities, find the possible values of:

(i) $\sin\alpha$ given $4\cos 2\alpha = \sin^2\alpha$.

Use $\cos 2\alpha = 1 - 2\sin^2\alpha$:

$$4(1 - 2\sin^2\alpha) = \sin^2\alpha$$

$$4 - 8\sin^2\alpha = \sin^2\alpha$$

$$9\sin^2\alpha = 4 \implies \sin\alpha = \pm\frac{2}{3}$$

$\sin\alpha = \pm\frac{2}{3}$

(ii) $\sec\beta$ given $2\tan^2\beta = 3 + 9\sec\beta$.

Use $\tan^2\beta = \sec^2\beta - 1$:

$$2(\sec^2\beta - 1) = 3 + 9\sec\beta$$

$$2\sec^2\beta - 9\sec\beta - 5 = 0$$

$$(2\sec\beta + 1)(\sec\beta - 5) = 0$$

$$\sec\beta = -\tfrac{1}{2} \text{ (rejected, since } |\sec\beta| \geq 1) \quad \text{or} \quad \sec\beta = 5$$

$\sec\beta = 5$

Question 11

(Jun 2014, Q2 – OCR 4723)

Worked Solution

Solve $5\cos 2\theta\theta = 2$ for $0° < \theta < 180°$.

Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\theta = \dfrac{1}{\sin\theta}$:

$$\frac{5(1-2\sin^2\theta)}{\sin\theta} = 2$$

$$5 - 10\sin^2\theta = 2\sin\theta$$

$$10\sin^2\theta + 2\sin\theta - 5 = 0$$

Quadratic formula:

$$\sin\theta = \frac{-2 \pm \sqrt{4+200}}{20} = \frac{-2 \pm \sqrt{204}}{20} = \frac{-1 \pm \sqrt{51}}{10}$$

Since $0° < \theta < 180°$, take the positive root:

$$\sin\theta = \frac{-1+\sqrt{51}}{10} \approx \frac{-1+7.1414}{10} \approx 0.6141$$

$$\theta = 37.9° \quad \text{or} \quad \theta = 142.1°$$

$\theta = 37.9°,\ 142.1°$

Question 12

(Jun 2016, Q4 – OCR 4723)

Worked Solution

It is given that A and B are angles such that $\sec^2 A - \tan A = 13$ and $\sin B \sec^2 B = 27 \cos B^2 B$.

Find the possible exact values of $\tan(A - B)$.

Finding $\tan A$**:** Use $\sec^2 A = 1 + \tan^2 A$:

$$1 + \tan^2 A - \tan A = 13 \implies \tan^2 A - \tan A - 12 = 0$$

$$(\tan A - 4)(\tan A + 3) = 0 \implies \tan A = 4 \text{ or } \tan A = -3$$

Finding $\tan B$**:** Rewrite as $\dfrac{\sin B}{\cos^2 B} = \dfrac{27 \cos B}{\sin^2 B}$:

$$\sin^3 B = 27 \cos^3 B \implies \tan^3 B = 27 \implies \tan B = 3$$

Case 1: $\tan A = 4$, $\tan B = 3$:

$$\tan(A - B) = \frac{4 - 3}{1 + 4 \times 3} = \frac{1}{13}$$

Case 2: $\tan A = -3$, $\tan B = 3$:

$$\tan(A - B) = \frac{-3 - 3}{1 + (-3)(3)} = \frac{-6}{-8} = \frac{3}{4}$$

$$\tan(A - B) = \frac{1}{13} \text{ or } \frac{3}{4}$$

Question 13

(Jun 2016, Q9 – OCR 4723)

Worked Solution

(i) Show that $\sin 2\theta(\tan\theta + \cot\theta) \equiv 2$.

$$\sin 2\theta(\tan\theta + \cot\theta) = 2\sin\theta\cos\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= 2\sin\theta\cos\theta \cdot \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = 2 \times 1 = 2 \qquad \checkmark$$

(ii)(a) Find the exact value of $\tan\frac{1}{12}\pi + \tan\frac{1}{8}\pi + \cot\frac{1}{12}\pi + \cot\frac{1}{8}\pi$.

Rearranging: $(\tan\frac{1}{12}\pi + \cot\frac{1}{12}\pi) + (\tan\frac{1}{8}\pi + \cot\frac{1}{8}\pi)$

From part (i) with $\theta = \frac{1}{12}\pi$: $\tan\frac{1}{12}\pi + \cot\frac{1}{12}\pi = \dfrac{2}{\sin\frac{1}{6}\pi} = \dfrac{2}{1/2} = 4$

With $\theta = \frac{1}{8}\pi$: $\tan\frac{1}{8}\pi + \cot\frac{1}{8}\pi = \dfrac{2}{\sin\frac{1}{4}\pi} = \dfrac{2}{1/\sqrt{2}} = 2\sqrt{2}$

$4 + 2\sqrt{2}$

(ii)(b) Solve $\sin 4\theta(\tan\theta + \cot\theta) = 1$ for $0 < \theta < \frac{1}{2}\pi$.

Use $\sin 4\theta = 2\sin 2\theta\cos 2\theta$:

$$2\sin 2\theta\cos 2\theta(\tan\theta + \cot\theta) = 1$$

Using the identity: $\sin 2\theta(\tan\theta + \cot\theta) = 2$, so:

$$2\cos 2\theta = 1 \implies \cos 2\theta = \frac{1}{4}$$

$$2\theta = \arccos\tfrac{1}{4} \approx 1.318 \implies \theta \approx 0.659 \text{ rad}$$

$\theta \approx 0.659$ rad

(ii)(c) Express $(1 - \cos 2\theta)^2(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta)^3$ in terms of $\sin\theta$.

$1 - \cos 2\theta = 2\sin^2\theta$

From the identity with $\frac{1}{2}\theta$: $\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta = \dfrac{2}{\sin\theta}$

$$(2\sin^2\theta)^2 \cdot \left(\frac{2}{\sin\theta}\right)^3 = 4\sin^4\theta \cdot \frac{8}{\sin^3\theta} = 32\sin\theta$$

$32\sin\theta$

Question 14

(Jun 2017, Q4 – OCR 4723)

Worked Solution

The angle θ, where $90^\circ < \theta < 180^\circ$, satisfies $3\sec^2\theta + 10\tan\theta = 11$.

(i) Find $\tan\theta$.

Use $\sec^2\theta = 1 + \tan^2\theta$:

$$3(1+\tan^2\theta) + 10\tan\theta = 11$$

$$3\tan^2\theta + 10\tan\theta - 8 = 0$$

$$(3\tan\theta - 2)(\tan\theta + 4) = 0$$

$$\tan\theta = \tfrac{2}{3} \text{ or } \tan\theta = -4$$

Since $90^\circ < \theta < 180^\circ$, $\tan\theta < 0$, so $\tan\theta = -4$.

$\tan\theta = -4$

(ii)(a) Find $\tan 2\theta$.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2(-4)}{1-16} = \frac{-8}{-15} = \frac{8}{15}$$

$\tan 2\theta = \dfrac{8}{15}$

(ii)(b) Find $\cot(2\theta + 135^\circ)$.

$$\tan(2\theta+135^\circ) = \frac{\tan 2\theta + \tan 135^\circ}{1-\tan 2\theta\tan 135^\circ} = \frac{\frac{8}{15}+(-1)}{1-\frac{8}{15}\cdot(-1)} = \frac{\frac{8}{15}-1}{1+\frac{8}{15}} = \frac{-\frac{7}{15}}{\frac{23}{15}} = -\frac{7}{23}$$

$$\cot(2\theta+135^\circ) = -\frac{23}{7}$$

$\cot(2\theta + 135^\circ) = -\dfrac{23}{7}$

End of Worked Solutions