Question 1

(Trig Identity & Equation)

Worked Solution

(a) Prove that $2\cot 2x + \tan x \equiv \cot x, \quad x \neq \frac{n\pi}{2},\ n \in \mathbb{Z}$.

$$2\cot 2x + \tan x = \frac{2}{\tan 2x} + \tan x$$
$$= \frac{2(1-\tan^2 x)}{2\tan x} + \tan x = \frac{1-\tan^2 x}{\tan x} + \tan x$$
$$= \frac{1-\tan^2 x + \tan^2 x}{\tan x} = \frac{1}{\tan x} = \cot x \quad \checkmark$$

(b) Solve $6\cot 2x + 3\tan x = \sec^2 x - 2$ for $-\pi \le x < \pi$.

Rewrite:

$$3(2\cot 2x + \tan x) = \sec^2 x - 2$$

Using part (a):

$$3\cot x = \sec^2 x - 2$$

Use $\sec^2 x = 1 + \tan^2 x$:

$$3\cot x = 1 + \tan^2 x - 2 = \tan^2 x - 1$$

Rewrite in terms of $\cot x$:

$$3\cot x = \cot^2 x - 1$$
$$\cot^2 x - 3\cot x - 1 = 0$$

Solve:

$$\cot x = \frac{3 \pm \sqrt{13}}{2}$$

$$\tan x = \frac{2}{3 \pm \sqrt{13}}$$

Numerically:

$$\tan x \approx 0.303 \Rightarrow x \approx 0.294,\ -2.848$$
$$\tan x \approx -3.303 \Rightarrow x \approx -1.277,\ 1.865$$

$x \approx -2.848,\ -1.277,\ 0.294,\ 1.865$

Question 2

Worked Solution

(a) Prove that $\sec 2A + \tan 2A \equiv \dfrac{\cos A + \sin A}{\cos A - \sin A}$, $A \neq \dfrac{(2n+1)\pi}{4}$, $n \in \mathbb{Z}$.

$$\sec 2A + \tan 2A = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} = \frac{1 + \sin 2A}{\cos 2A}$$

Use $\sin 2A = 2\sin A \cos A$ and $\cos 2A = \cos^2 A - \sin^2 A$:

$$= \frac{1 + 2\sin A \cos A}{\cos^2 A - \sin^2 A} = \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)} = \frac{\cos A + \sin A}{\cos A - \sin A} \qquad \checkmark$$

(b) Solve $\sec 2\theta + \tan 2\theta = \dfrac{1}{2}$ for $0 \leq \theta < 2\pi$.

From part (a):

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1}{2}$$

$$2(\cos\theta + \sin\theta) = \cos\theta - \sin\theta$$

$$2\cos\theta + 2\sin\theta = \cos\theta - \sin\theta$$

$$\cos\theta = -3\sin\theta \implies \tan\theta = -\frac{1}{3}$$

$$\theta = \pi - \arctan\tfrac{1}{3} \approx 2.820 \quad \text{or} \quad \theta = 2\pi - \arctan\tfrac{1}{3} \approx 5.961$$

$\theta \approx 2.820,\ 5.961$

Question 3

Worked Solution

(a) Prove that $1 - \cos 2\theta \equiv \tan\theta \sin 2\theta$, $\theta \neq \dfrac{(2n+1)\pi}{2}$, $n \in \mathbb{Z}$.

Starting from the right-hand side:

$$\tan\theta\sin 2\theta = \frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta = 2\sin^2\theta = 1-\cos 2\theta \qquad \checkmark$$

(b) Solve $(\sec^2 x - 5)(1-\cos 2x) = 3\tan^2 x\sin 2x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Using the result from (a), $1 - \cos 2x = \tan x \sin 2x$:

$$(\sec^2 x - 5)\tan x\sin 2x = 3\tan^2 x\sin 2x$$

Either $\sin 2x = 0$ (giving $x = 0$), or divide through by $\sin 2x$:

$$(\sec^2 x - 5)\tan x = 3\tan^2 x$$

Either $\tan x = 0$ (giving $x = 0$), or divide by $\tan x$:

$$\sec^2 x - 5 = 3\tan x$$

Use $\sec^2 x = 1 + \tan^2 x$:

$$1 + \tan^2 x - 5 = 3\tan x$$
$$\tan^2 x - 3\tan x - 4 = 0$$
$$(\tan x - 4)(\tan x + 1) = 0$$
$$\tan x = 4 \implies x \approx 1.326 \qquad \tan x = -1 \implies x = -\frac{\pi}{4}$$

$x = 0,\ -\dfrac{\pi}{4},\ 1.326$

Question 4

Worked Solution

(i) Solve $4\sin x = \sec x$ for $0 \le x < \dfrac{\pi}{2}$.

$$4\sin x = \frac{1}{\cos x} \implies 4\sin x\cos x = 1 \implies 2\sin 2x = 1 \implies \sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{6} \quad \text{or} \quad 2x = \frac{5\pi}{6} \implies x = \frac{\pi}{12}, \frac{5\pi}{12}$$

$x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

(ii) Solve $5\sin\theta - 5\cos\theta = 2$ for $0 \le \theta < 360°$.

Express as $R\sin(\theta - \alpha)$: $R = \sqrt{5^2 + 5^2} = 5\sqrt{2}$, $\tan\alpha = 1 \Rightarrow \alpha = 45°$

$$5\sqrt{2}\sin(\theta - 45°) = 2 \implies \sin(\theta - 45°) = \frac{2}{5\sqrt{2}} = \frac{\sqrt{2}}{5}$$

$$\theta - 45° = \arcsin\frac{\sqrt{2}}{5} \approx 16.4° \quad \text{or} \quad 180° - 16.4° = 163.6°$$

$$\theta \approx 61.4° \quad \text{or} \quad \theta \approx 208.6°$$

$\theta \approx 61.4°,\ 208.6°$

Question 5

Worked Solution

(a) Show that $\csc 2x + \cot 2x \equiv \cot x$, $x \neq 90n°$, $n \in \mathbb{R}$.

$$\csc 2x + \cot 2x = \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} = \frac{1 + \cos 2x}{\sin 2x}$$

Use $1 + \cos 2x = 2\cos^2 x$ and $\sin 2x = 2 \sin x \cos x$:

$$= \frac{2\cos^2 x}{2 \sin x \cos x} = \frac{\cos x}{\sin x} = \cot x \qquad \checkmark$$

(b) Solve $\csc(4\theta + 10°) + \cot(4\theta + 10°) = \sqrt{3}$ for $0 \leq \theta < 180°$.

Using the result from (a) with $x = 2\theta + 5°$:

$$\cot(2\theta + 5°) = \sqrt{3}$$

$$\tan(2\theta + 5°) = \frac{1}{\sqrt{3}} \implies 2\theta + 5° = 30° \implies \theta = 12.5°$$

Next solution: $2\theta + 5° = 30° + 180° = 210° \implies \theta = 102.5°$

$\theta = 12.5°,\ 102.5°$

Question 6

Worked Solution

(a) Prove that $\sin 2x - \tan x \equiv \tan x \cos 2x$, $x \neq (2n+1)90°$, $n \in \mathbb{Z}$.

$$\sin 2x - \tan x = 2\sin x\cos x - \frac{\sin x}{\cos x} = \frac{2\sin x\cos^2 x - \sin x}{\cos x} = \frac{\sin x(2\cos^2 x - 1)}{\cos x}$$

$$= \frac{\sin x}{\cos x} \cdot (2\cos^2 x - 1) = \tan x \cos 2x \qquad \checkmark$$

(b) Solve $\sin 2x - \tan x = 3\tan x \sin x$ for $0 \leq x < 360°$.

Using the identity:

$$\tan x \cos 2x = 3\tan x \sin x$$

$$\tan x(\cos 2x - 3\sin x) = 0$$

Case 1: $\tan x = 0 \Rightarrow x = 0°,\ 180°$

Case 2: $\cos 2x = 3\sin x$. Use $\cos 2x = 1 - 2\sin^2 x$:

$$1 - 2\sin^2 x = 3\sin x \implies 2\sin^2 x + 3\sin x - 1 = 0$$

$$\sin x = \frac{-3 \pm \sqrt{9+8}}{4} = \frac{-3 \pm \sqrt{17}}{4}$$

Taking the positive root: $\sin x = \dfrac{-3+\sqrt{17}}{4} \approx 0.2808 \Rightarrow x \approx 16.3°,\ 163.7°$

(Negative root: $\sin x \approx -1.78$ – rejected)

$x = 0°,\ 16.3°,\ 163.7°,\ 180°$

Question 7

Worked Solution

(i) Show $\csc 2x = \lambda \csc x \sec x$ and state λ.

$$\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x\cos x} = \frac{1}{2}\cdot\frac{1}{\sin x}\cdot\frac{1}{\cos x} = \frac{1}{2}\csc x\sec x$$

$$\lambda = \frac{1}{2}$$

(ii) Solve $3\sec^2\theta + 3\sec\theta = 2\tan^2\theta$ for $0 \le \theta < 2\pi$. Give answers in terms of π.

Use $\tan^2\theta = \sec^2\theta - 1$:

$$3\sec^2\theta + 3\sec\theta = 2(\sec^2\theta - 1)$$

$$\sec^2\theta + 3\sec\theta + 2 = 0$$

$$(\sec\theta + 2)(\sec\theta + 1) = 0$$

$$\sec\theta = -2 \implies \cos\theta = -\tfrac{1}{2} \implies \theta = \frac{2\pi}{3},\ \frac{4\pi}{3}$$

$$\sec\theta = -1 \implies \cos\theta = -1 \implies \theta = \pi$$

$$\theta = \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3}$$

Question 8

Worked Solution

(a) Prove that $\dfrac{1}{\sin 2\theta} - \dfrac{\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$, $\theta \neq 90n^\circ$, $n \in \mathbb{Z}$.

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad \checkmark$$

(b)(i) Show that $\tan 15^\circ = 2 - \sqrt{3}$.

Using the result with $\theta = 15^\circ$ (so $2\theta = 30^\circ$):

$$\tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3} \qquad \checkmark$$

(b)(ii) Solve $\csc 4x - \cot 4x = 1$ for $0^\circ < x < 360^\circ$.

From the identity with $\theta = 2x$:

$$\tan 2x = 1 \implies 2x = 45^\circ + 180^\circ k$$

$$2x = 45^\circ,\ 225^\circ,\ 405^\circ,\ 585^\circ$$

$$x = 22.5^\circ,\ 112.5^\circ,\ 202.5^\circ,\ 292.5^\circ$$

$x = 22.5^\circ,\ 112.5^\circ,\ 202.5^\circ,\ 292.5^\circ$

Question 9

Worked Solution

Find all solutions of $2\cos 2\theta = 1 - 2\sin\theta$ in the interval $0 \leq \theta < 360°$.

Use $\cos 2\theta = 1 - 2\sin^2\theta$:

$$2(1 - 2\sin^2\theta) = 1 - 2\sin\theta$$
$$2 - 4\sin^2\theta = 1 - 2\sin\theta$$
$$4\sin^2\theta - 2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

$\sin\theta = \dfrac{1+\sqrt{5}}{4} \approx 0.809$: $\theta = 54°$ or $\theta = 126°$

$\sin\theta = \dfrac{1-\sqrt{5}}{4} \approx -0.309$: $\theta = 198°$ or $\theta = 342°$

$\theta = 54°,\ 126°,\ 198°,\ 342°$

Question 10

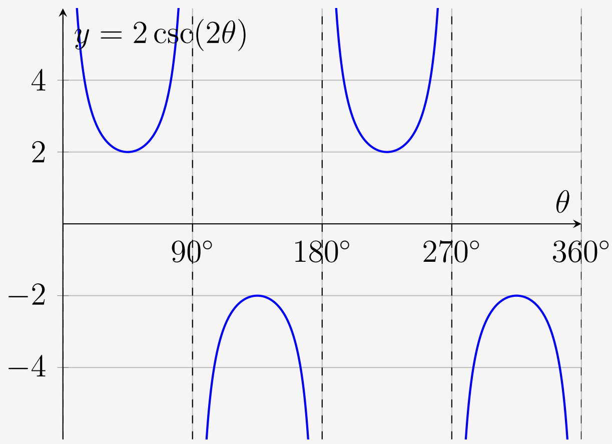
Worked Solution

(a) Prove that $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} \equiv 2\csc 2\theta$, $\theta \neq 90n°$.

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta} = \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\csc 2\theta \qquad \checkmark$$

(b) Sketch $y = 2\csc 2\theta$ for $0° < \theta < 360°$.

The graph has vertical asymptotes at $\theta = 0°, 90°, 180°, 270°, 360°$, with turning points at $y = \pm 2$ and four branches (two positive, two negative).



(c) Solve $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = 3$ for $0° < \theta < 360°$.

From (a): $2\csc 2\theta = 3 \Rightarrow \sin 2\theta = \dfrac{2}{3}$

$$2\theta = 41.8°,\ 138.2°,\ 401.8°,\ 498.2°$$

$$\theta = 20.9°,\ 69.1°,\ 200.9°,\ 249.1°$$

$\theta = 20.9°,\ 69.1°,\ 200.9°,\ 249.1°$

End of Worked Solutions