Question 1

Worked Solution

Claim: Given a rational number a and an irrational number b, $a - b$ is irrational.

Step 1: Assume the opposite (the contradiction hypothesis).

Assume, for the sake of contradiction, that $a - b$ is *rational*.

Step 2: Set up algebraic expressions for each quantity.

Since a is rational, write $a = \frac{m}{n}$ for some integers m, n with $n \neq 0$.

Since we are assuming $a - b$ is rational, write $a - b = \frac{p}{q}$ for some integers p, q with $q \neq 0$.

Step 3: Derive a consequence.

Substituting $a = \frac{m}{n}$ into $a - b = \frac{p}{q}$:

$$\frac{m}{n} - b = \frac{p}{q}$$

Rearranging to make b the subject:

$$b = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$$

Step 4: Identify the contradiction.

Since m, n, p, q are all integers and $nq \neq 0$, the expression $\frac{mq - pn}{nq}$ is a ratio of two integers — i.e. it is *rational*.

But this contradicts the assumption that b is irrational.

The assumption that $a - b$ is rational leads to a contradiction. Therefore $a - b$ must be **irrational**. ■

Question 2

Worked Solution

Claim: The product of two odd numbers is odd.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that there exist two odd numbers whose product is *even*.

Step 2: Write general expressions for odd numbers.

Let the two odd numbers be $2m + 1$ and $2n + 1$, where m, n are integers.

(Any integer of the form $2k + 1$ is odd by definition, since it leaves remainder 1 on division by 2.)

Step 3: Compute their product.

$$\begin{aligned}(2m+1)(2n+1) &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1\end{aligned}$$

Step 4: Identify the contradiction.

Since m and n are integers, $2mn + m + n$ is also an integer. Therefore $2(2mn + m + n)$ is even, and so $2(2mn + m + n) + 1$ is *odd*.

This contradicts our assumption that that there exist two odd numbers whose product is *even*.

This is a contradiction. Therefore the product of two odd numbers is always **odd**. ■

Question 3

Worked Solution

Claim: If n is odd, then $n^3 + 1$ is even.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that there exists an integer n such that n is *odd* and $n^3 + 1$ is also *odd*.

Step 2: Write n in terms of a general odd number.

Let $n = 2k + 1$ for some integer k.

Step 3: Compute $n^3 + 1$.

$$\begin{aligned}(2k+1)^3 + 1 &= \left(8k^3 + 12k^2 + 6k + 1\right) + 1 \\ &= 8k^3 + 12k^2 + 6k + 2 \\ &= 2\left(4k^3 + 6k^2 + 3k + 1\right)\end{aligned}$$

Step 4: Identify the contradiction.

Since k is an integer, $4k^3 + 6k^2 + 3k + 1$ is an integer, so $n^3 + 1 = 2(\ldots)$ is *even*.

But we assumed $n^3 + 1$ is odd — a direct contradiction.

The assumption leads to a contradiction. Therefore, if n is odd, $n^3 + 1$ is **even**. ■

Question 4

Part (a)

Worked Solution

Claim: If n^2 is an even integer, then n is also even.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that there exists a number n such that n^2 is even but n is *odd*.

Step 2: Write n as a general odd number.

Let $n = 2k + 1$ for some integer k.

Step 3: Compute n^2.

$$\begin{aligned} n^2 = (2k+1)^2 &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Step 4: Identify the contradiction.

Since $2k^2 + 2k$ is an integer, $n^2 = 2(2k^2 + 2k) + 1$ is *odd*.

But we assumed n^2 is even — a contradiction.

The assumption leads to a contradiction. Therefore if n^2 is even, n must be **even**. ■

Part (b)

Worked Solution

Claim: $\sqrt{2}$ is irrational.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that $\sqrt{2}$ is *rational*.

Step 2: Write $\sqrt{2}$ as a fraction in lowest terms.

Then $\sqrt{2} = \frac{a}{b}$ for some integers a and b, where a and b have *no common factors* (i.e. the fraction is fully simplified).

Step 3: Show a must be even.

Squaring both sides:

$$2 = \frac{a^2}{b^2} \implies a^2 = 2b^2$$

Since $2b^2$ is even, a^2 is even. By part (a), this means a must be even.

Step 4: Substitute $a = 2c$ and show b must also be even.

Since a is even, write $a = 2c$ for some integer c. Substituting:

$$(2c)^2 = 2b^2 \implies 4c^2 = 2b^2 \implies b^2 = 2c^2$$

Since $2c^2$ is even, b^2 is even, and by part (a) again, b must be even.

Step 5: Identify the contradiction.

Both a and b are even, so they share a common factor of 2. But we assumed a and b have no common factors.

This is a contradiction. Therefore $\sqrt{2}$ is **irrational**. ■

Question 5

Worked Solution

Claim: There is no greatest positive rational number.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that there *does* exist a greatest positive rational number; call it $\frac{a}{b}$, where a, b are positive integers.

Step 2: Construct a rational number that is larger.

Consider the number $\frac{a}{b} + 1$. Since $\frac{a}{b} > 0$, this number is clearly greater than $\frac{a}{b}$.

Step 3: Show this new number is rational.

$$\frac{a}{b} + 1 = \frac{a}{b} + \frac{b}{b} = \frac{a+b}{b}$$

Since a and b are integers and $b \neq 0$, $\frac{a+b}{b}$ is by definition *rational* since the numerator and denominator are integers.

Step 4: Identify the contradiction.

We have found a rational number $\frac{a+b}{b}$ that is strictly greater than $\frac{a}{b}$. But $\frac{a}{b}$ was assumed to be the greatest positive rational number.

This is a contradiction. Therefore there is **no greatest positive rational number**. ■

Question 6

Worked Solution

Claim: There exist no integers a and b for which $25a + 15b = 1$.

Step 1: Assume the opposite.

Assume, for the sake of contradiction, that there *do* exist integers a and b such that $25a + 15b = 1$.

Step 2: Derive a consequence by dividing by 5.

Both 25 and 15 are multiples of 5, so divide both sides of $25a + 15b = 1$ by 5:

$$5a + 3b = \frac{1}{5}$$

Step 3: Identify the contradiction.

Since a and b are integers:

- $5a$ is an integer (an integer multiplied by an integer),
- $3b$ is an integer,
- therefore $5a + 3b$ is an integer.

But $\frac{1}{5}$ is *not* an integer. So we have an integer equal to a non-integer — a contradiction.

The assumption leads to a contradiction. Therefore there exist **no integers** a and b for which $25a + 15b = 1$. ■

End of Worked Solutions