Question 1

Worked Solution

Express $\dfrac{5x+3}{(2x+1)(x+1)^2}$ in partial fractions.

$$\frac{5x+3}{(2x+1)(x+1)^2} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
$$5x+3 \equiv A(x+1)^2 + B(2x+1)(x+1) + C(2x+1)$$

Set $x = -1$: $-5+3 = -2 = C(2(-1)+1) = C(-1) \implies C = 2.$

Set $x = -\frac{1}{2}$: $-\frac{5}{2} + 3 = \frac{1}{2} = A\left(\frac{1}{2}\right)^2 = \frac{A}{4} \implies A = 2.$

Compare x^2 coefficients: $0 = A + 2B \implies B = -\dfrac{A}{2} = -1.$

Check constant terms: $3 = A + B + C = 2 - 1 + 2 = 3$ ✓

$A = 2,\ \ B = -1,\ \ C = 2$

$$\frac{5x+3}{(2x+1)(x+1)^2} = \frac{2}{2x+1} - \frac{1}{x+1} + \frac{2}{(x+1)^2}$$

Question 2

Worked Solution

$$\frac{9x^2}{(x-1)^2(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+1}$$

Find A, B and C.

$$9x^2 \equiv A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$$

Set $x = 1$: $9 = B(3) \implies B = 3$.

Set $x = -\frac{1}{2}$: $\frac{9}{4} = C\left(-\frac{3}{2}\right)^2 = \frac{9C}{4} \implies C = 1$.

Compare x^2 coefficients: $9 = 2A + C \implies 2A = 8 \implies A = 4$.

Check x terms: $0 = A \cdot (-1 \cdot 2 + 1 \cdot 1) + B \cdot 2 + C \cdot (-2) = A(-1) + 6 - 2 = -4 + 4 = 0$ ✓

$A = 4, \ B = 3, \ C = 1$

$$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{4}{x-1} + \frac{3}{(x-1)^2} + \frac{1}{2x+1}$$

Question 3

Worked Solution

Given $\dfrac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \dfrac{dx + e}{x^2 - 4}$, $x \neq \pm 2$, find a, b, c, d, e.

Perform polynomial long division of $3x^4 - 2x^3 - 5x^2 + 0x - 4$ by $x^2 - 4$ (i.e. $x^2 + 0x - 4$):

$$
\begin{array}{r}
3x^2 - 2x + 7 \\
x^2 - 4 \,\big)\, \overline{3x^4 - 2x^3 - 5x^2 + 0x - 4} \\
\underline{3x^4 \qquad\quad - 12x^2} \\
-2x^3 + 7x^2 + 0x \\
\underline{-2x^3 \qquad\quad + 8x} \\
7x^2 - 8x - 4 \\
\underline{7x^2 \qquad - 28} \\
-8x + 24
\end{array}
$$

So:

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} = 3x^2 - 2x + 7 + \frac{-8x + 24}{x^2 - 4}$$

$a = 3,\ b = -2,\ c = 7,\ d = -8,\ e = 24$

Question 4

Worked Solution

$\dfrac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$

Find A, B and C.

Since the numerator is degree 2 and the denominator is degree 2, the integer part A can be found first.

$(x-1)(x+2) = x^2 + x - 2$.

$2x^2 + 5x - 10 = 2(x^2 + x - 2) + 3x - 6 = 2(x-1)(x+2) + 3(x-2)$.

So:

$$\frac{2x^2+5x-10}{(x-1)(x+2)} = 2 + \frac{3x-6}{(x-1)(x+2)}$$

Wait – let me verify via substitution directly.

$2x^2 + 5x - 10 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)$

Set $x = 1$: $2 + 5 - 10 = -3 = 3B \implies B = -1$.

Set $x = -2$: $8 - 10 - 10 = -12 = -3C \implies C = 4$.

Compare x^2: $2 = A \implies A = 2$.

Check: $A = 2$, $B = -1$, $C = 4$: constant $= -2A - 2B + C = -4 + 2 + 4 = 2$... but LHS constant $= -10$.

Let me redo: $A(x-1)(x+2) + B(x+2) + C(x-1)$, constant term $= A(-1)(2) + B(2) + C(-1) = -2A + 2B - C$.

$-2(2) + 2(-1) - 4 = -4 - 2 - 4 = -10$ ✓

$A = 2, \;\; B = -1, \;\; C = 4$

$$\frac{2x^2+5x-10}{(x-1)(x+2)} = 2 - \frac{1}{x-1} + \frac{4}{x+2}$$

Question 5

Worked Solution

$\mathrm{f}(x) = \dfrac{4-2x}{(2x+1)(x+1)(x+3)} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

Find A, B and C.

$$4-2x \equiv A(x+1)(x+3) + B(2x+1)(x+3) + C(2x+1)(x+1)$$

Set $x = -\frac{1}{2}$**:** $4+1 = 5 = A\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) = \frac{5A}{4} \implies A = 4.$

Set $x = -1$**:** $4+2 = 6 = B(-1)(2) = -2B \implies B = -3.$

Set $x = -3$**:** $4+6 = 10 = C(-5)(-2) = 10C \implies C = 1.$

$A = 4,\ \ B = -3,\ \ C = 1$

$$\mathrm{f}(x) = \frac{4}{2x+1} - \frac{3}{x+1} + \frac{1}{x+3}$$

Question 6

Worked Solution

$\dfrac{1 + 11x - 6x^2}{(x-3)(1-2x)} \equiv A + \dfrac{B}{x-3} + \dfrac{C}{1-2x}$

Find A, B and C.

The numerator is degree 2 and denominator is degree 2 (leading term $-2x^2$), so A is the ratio of leading coefficients: $\dfrac{-6}{-2} = 3$.

So $A = 3$.

$$1 + 11x - 6x^2 \equiv 3(x-3)(1-2x) + B(1-2x) + C(x-3)$$

Set $x = 3$**:** $1 + 33 - 54 = -20 = B(1-6) = -5B \implies B = 4.$

Set $x = \frac{1}{2}$**:** $1 + \frac{11}{2} - \frac{3}{2} = 1 + 4 = 5 = C\left(\frac{1}{2} - 3\right) = -\frac{5}{2}C \implies C = -2.$

$A = 3,\ \ B = 4,\ \ C = -2$

$$\frac{1 + 11x - 6x^2}{(x-3)(1-2x)} = 3 + \frac{4}{x-3} - \frac{2}{1-2x}$$

Question 7

Worked Solution

Express $\dfrac{9x^2+20x-10}{(x+2)(3x-1)}$ in partial fractions.

The numerator is degree 2 and denominator $(x+2)(3x-1) = 3x^2+5x-2$ is degree 2. The ratio of leading coefficients is $\dfrac{9}{3} = 3$, so the integer part is 3.

$9x^2+20x-10 = 3(3x^2+5x-2) + r(x)$

$3(3x^2+5x-2) = 9x^2+15x-6$

$r(x) = (9x^2+20x-10) - (9x^2+15x-6) = 5x-4$

So:

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} = 3 + \frac{5x-4}{(x+2)(3x-1)}$$

Decompose $\dfrac{5x-4}{(x+2)(3x-1)} = \dfrac{B}{x+2} + \dfrac{C}{3x-1}$:

$5x-4 = B(3x-1) + C(x+2)$

Set $x = -2$**:** $-14 = -7B \implies B = 2$.

Set $x = \frac{1}{3}$**:** $\frac{5}{3} - 4 = -\frac{7}{3} = C \cdot \frac{7}{3} \implies C = -1$.

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} = 3 + \frac{2}{x+2} - \frac{1}{3x-1}$$

End of Worked Solutions