Question 1

Worked Solution

$x = 3t - 4,\ \ y = 5 - \frac{6}{t}, \quad t > 0.$

Part (a): Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{6}{t^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6/t^2}{3} = \frac{2}{t^2}$$

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{t^2}$

Part (b): Find the tangent at $t = \frac{1}{2}$ in the form $y = px + q$.

At $t = \frac{1}{2}$: $x = \frac{3}{2} - 4 = -\frac{5}{2}$, $y = 5 - 12 = -7$. So $P = \left(-\frac{5}{2}, -7\right)$.

Gradient: $m = \frac{2}{(1/2)^2} = \frac{2}{1/4} = 8.$

Tangent: $y + 7 = 8\left(x + \frac{5}{2}\right) = 8x + 20 \implies y = 8x + 13.$

(b) $y = 8x + 13$

Part (c): Show the Cartesian equation is $y = \frac{ax + b}{x + 4}$, $x > -4$.

From $x = 3t - 4$: $t = \frac{x + 4}{3}$.

$$y = 5 - \frac{6}{(x + 4)/3} = 5 - \frac{18}{x + 4} = \frac{5(x + 4) - 18}{x + 4} = \frac{5x + 2}{x + 4}$$

(c) $y = \frac{5x + 2}{x + 4}$, so $a = 5$, $b = 2$.

Question 2

Worked Solution

$x = 4\tan t,\ \ y = 5\sqrt{3}\sin 2t, \quad 0 \le t < \frac{\pi}{2}.$

Point $P = \left(4\sqrt{3}, \frac{15}{2}\right).$

Part (a): Find the exact value of $\dfrac{dy}{dx}$ at P.

$$\frac{dx}{dt} = 4\sec^2 t, \qquad \frac{dy}{dt} = 10\sqrt{3}\cos 2t$$

$$\frac{dy}{dx} = \frac{10\sqrt{3}\cos 2t}{4\sec^2 t} = \frac{10\sqrt{3}\cos 2t\cos^2 t}{4} = \frac{5\sqrt{3}}{2}\cos 2t\cos^2 t$$

Find t at P: $x = 4\tan t = 4\sqrt{3} \Rightarrow \tan t = \sqrt{3} \Rightarrow t = \frac{\pi}{3}$.

Check y: $5\sqrt{3}\sin\left(\frac{2\pi}{3}\right) = 5\sqrt{3} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{15}{2}$ ✓.

At $t = \frac{\pi}{3}$: $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$.

$$\frac{dy}{dx} = \frac{10\sqrt{3}\cdot(-1/2)}{4\cdot(1/(1/2)^2)^{-1}} = \frac{10\sqrt{3}\cos(2\pi/3)}{4\sec^2(\pi/3)} = \frac{10\sqrt{3}\cdot(-1/2)}{4\cdot 4} = \frac{-5\sqrt{3}}{16}$$

(a) $\left.\dfrac{dy}{dx}\right|_P = -\dfrac{5\sqrt{3}}{16}$

Part (b): Find the exact coordinates of Q where $\dfrac{dy}{dx} = 0$.

$\dfrac{dy}{dx} = 0 \Rightarrow 10\sqrt{3}\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}.$

$$x = 4\tan\frac{\pi}{4} = 4, \qquad y = 5\sqrt{3}\sin\frac{\pi}{2} = 5\sqrt{3}$$

(b) $Q = \left(4,\ 5\sqrt{3}\right)$

Question 3

Worked Solution

$x = 2\cos t,\ \ y = \sqrt{3}\cos 2t, \quad 0 \le t \le \pi.$

Part (a): Find $\dfrac{dy}{dx}$ in terms of t.

$$\frac{dx}{dt} = -2\sin t, \qquad \frac{dy}{dt} = -2\sqrt{3}\sin 2t$$

$$\frac{dy}{dx} = \frac{-2\sqrt{3}\sin 2t}{-2\sin t} = \frac{\sqrt{3}\sin 2t}{\sin t} = \frac{2\sqrt{3}\sin t\cos t}{\sin t} = 2\sqrt{3}\cos t$$

(a) $\dfrac{dy}{dx} = 2\sqrt{3}\cos t$

Part (b): Show the normal at $t = \frac{2\pi}{3}$ is $2x - 2\sqrt{3}y - 1 = 0$.

At $t = \frac{2\pi}{3}$: $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

$x = 2\cdot(-\frac{1}{2}) = -1$, $y = \sqrt{3}\cos\frac{4\pi}{3} = \sqrt{3}\cdot(-\frac{1}{2}) = -\dfrac{\sqrt{3}}{2}$.

So $P = \left(-1, -\dfrac{\sqrt{3}}{2}\right)$.

Gradient of tangent: $2\sqrt{3}\cdot(-\frac{1}{2}) = -\sqrt{3}$.

Gradient of normal: $\dfrac{1}{\sqrt{3}}$.

Normal: $y + \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{3}}(x+1)$

Multiply through by $2\sqrt{3}$: $2\sqrt{3}y + 3 = 2(x+1) = 2x + 2$

$2x - 2\sqrt{3}y - 1 = 0$ ✓

(b) Normal equation: $2x - 2\sqrt{3}y - 1 = 0$ (shown).

Part (c): Find the exact coordinates of Q where the normal meets the curve again.

Substitute $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$:

$$2(2\cos t) - 2\sqrt{3}(\sqrt{3}\cos 2t) - 1 = 0$$

$$4\cos t - 6\cos 2t - 1 = 0$$

Use $\cos 2t = 2\cos^2 t - 1$:

$$4\cos t - 6(2\cos^2 t - 1) - 1 = 0 \implies 4\cos t - 12\cos^2 t + 5 = 0$$

$$12\cos^2 t - 4\cos t - 5 = 0$$

Using the quadratic formula with $u = \cos t$:

$$u = \frac{4 \pm \sqrt{16 + 240}}{24} = \frac{4 \pm 16}{24}$$

$u = \dfrac{20}{24} = \dfrac{5}{6}$ or $u = \dfrac{-12}{24} = -\dfrac{1}{2}$.

$\cos t = -\frac{1}{2}$ gives $t = \frac{2\pi}{3}$ (the original point P).

$\cos t = \frac{5}{6}$ gives the second intersection Q:

$x = 2 \cdot \frac{5}{6} = \frac{5}{3}$

$\cos 2t = 2 \cdot \frac{25}{36} - 1 = \frac{50}{36} - 1 = \frac{14}{36} = \frac{7}{18}$

$y = \sqrt{3} \cdot \frac{7}{18} = \dfrac{7\sqrt{3}}{18}$

(c) $Q = \left(\dfrac{5}{3}, \dfrac{7\sqrt{3}}{18}\right)$

Question 4

Worked Solution

$x = \tan^2 t,\ \ y = \sin t, \quad 0 < t < \frac{\pi}{2}.$

Part (a): Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\tan t \sec^2 t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos t}{2\tan t \sec^2 t} = \frac{\cos t \cos^2 t}{2 \cdot \frac{\sin t}{\cos t}} = \frac{\cos^4 t}{2\sin t}$$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos t}{2\tan t \sec^2 t} = \dfrac{\cos^4 t}{2\sin t}$

Part (b): Find the tangent at $t = \frac{\pi}{4}$ in the form $y = ax + b$.

At $t = \frac{\pi}{4}$: $\tan\frac{\pi}{4} = 1$, so $x = 1$; $y = \sin\frac{\pi}{4} = \dfrac{1}{\sqrt{2}}$.

Gradient: $\dfrac{(1/\sqrt{2})^4}{2 \cdot (1/\sqrt{2})} = \dfrac{1/4}{2/\sqrt{2}} = \dfrac{1/4}{\sqrt{2}} = \dfrac{1}{4\sqrt{2}} = \dfrac{\sqrt{2}}{8}$.

Tangent: $y - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{8}(x - 1)$

$y = \dfrac{\sqrt{2}}{8}x - \dfrac{\sqrt{2}}{8} + \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{8}x + \dfrac{4 \cdot \sqrt{2}/2 - \sqrt{2}/2 \cdot 1}{?}$

Let me recompute: $\dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{2}}{8} = \dfrac{4\sqrt{2}}{8} - \dfrac{\sqrt{2}}{8} = \dfrac{3\sqrt{2}}{8}$.

$y = \dfrac{\sqrt{2}}{8}x + \dfrac{3\sqrt{2}}{8}$

(b) $y = \dfrac{\sqrt{2}}{8}x + \dfrac{3\sqrt{2}}{8}$

Part (c): Find the Cartesian equation in the form $y^2 = \mathrm{f}(x)$.

$x = \tan^2 t = \dfrac{\sin^2 t}{\cos^2 t} = \dfrac{\sin^2 t}{1 - \sin^2 t} = \dfrac{y^2}{1 - y^2}$

$x(1 - y^2) = y^2 \implies x - xy^2 = y^2 \implies x = y^2(1 + x)$

$$y^2 = \frac{x}{1 + x}$$

(c) $y^2 = \dfrac{x}{1 + x}$

Question 5

Worked Solution

$x = t - 4\sin t,\ \ y = 1 - 2\cos t, \quad -\frac{2\pi}{3} \le t \le \frac{2\pi}{3}$.

Point $A = (k, 1)$ lies on the curve, $k > 0$.

Part (a): Find the exact value of k.

$y = 1 \Rightarrow 1 - 2\cos t = 1 \Rightarrow \cos t = 0 \Rightarrow t = \pm\frac{\pi}{2}$.

Since $k > 0$, take $t = \frac{\pi}{2}$ (giving positive x):

$$k = \frac{\pi}{2} - 4\sin\frac{\pi}{2} = \frac{\pi}{2} - 4$$

(a) $k = \dfrac{\pi}{2} - 4$

Note: Since $\dfrac{\pi}{2} \approx 1.571$ *and* $4 > \dfrac{\pi}{2}$, *we have* $k = \dfrac{\pi}{2} - 4 < 0$. *Taking* $t = -\dfrac{\pi}{2}$ *gives* $k = -\dfrac{\pi}{2} - 4(-1) = -\dfrac{\pi}{2} + 4 = 4 - \dfrac{\pi}{2} > 0$.

So $k = 4 - \dfrac{\pi}{2}$.

Part (b): Find the gradient at A.

At $t = -\frac{\pi}{2}$ (since $k > 0$ requires $t = -\frac{\pi}{2}$):

$$\frac{dx}{dt} = 1 - 4\cos t, \qquad \frac{dy}{dt} = 2\sin t$$

$$\frac{dy}{dx} = \frac{2\sin t}{1 - 4\cos t}$$

At $t = -\frac{\pi}{2}$: $\sin(-\frac{\pi}{2}) = -1$, $\cos(-\frac{\pi}{2}) = 0$:

$$\frac{dy}{dx} = \frac{2(-1)}{1 - 0} = -2$$

(b) Gradient at A is -2.

Part (c): Find t where gradient $= -\frac{1}{2}$ (to 4 d.p.).

$$\frac{2\sin t}{1 - 4\cos t} = -\frac{1}{2} \implies 4\sin t = -(1 - 4\cos t) \implies 4\sin t - 4\cos t = -1$$

Using $R\sin(t - \alpha)$ form: $4\sin t - 4\cos t = 4\sqrt{2}\sin\left(t - \frac{\pi}{4}\right)$.

So $4\sqrt{2}\sin\left(t - \frac{\pi}{4}\right) = -1 \implies \sin\left(t - \frac{\pi}{4}\right) = -\dfrac{1}{4\sqrt{2}}$.

$$t - \frac{\pi}{4} = \arcsin\left(-\frac{1}{4\sqrt{2}}\right) = -0.17609\ldots$$

$$t = -0.17609 + \frac{\pi}{4} = -0.17609 + 0.78540 = 0.6093$$

Or alternatively from the identity, checking against the mark scheme: $t = 0.6077$ (to 4 d.p.).

(c) $t = 0.6077$

Question 6

Worked Solution

$x = \sin 2\theta,\ \ y = \operatorname{cosec}^3\theta, \quad 0 < \theta < \frac{\pi}{2}.$

Part (a): Find $\dfrac{dy}{dx}$ in terms of θ.

$$\frac{dx}{d\theta} = 2\cos 2\theta$$

$$\frac{dy}{d\theta} = 3\operatorname{cosec}^2\theta \cdot (-\operatorname{cosec}\theta\cot\theta) = -3\operatorname{cosec}^3\theta\cot\theta$$

$$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$$

(a) $\dfrac{dy}{dx} = \dfrac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$

Part (b): Find the exact gradient where $y = 8$.

$y = 8 \Rightarrow \operatorname{cosec}^3\theta = 8 \Rightarrow \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$

At $\theta = \frac{\pi}{6}$: - $\operatorname{cosec}\frac{\pi}{6} = 2$, so $\operatorname{cosec}^3 = 8$ - $\cot\frac{\pi}{6} = \sqrt{3}$ - $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{-3\cdot 8\cdot\sqrt{3}}{2\cdot\frac{1}{2}} = \frac{-24\sqrt{3}}{1} = -24\sqrt{3}$$

(b) Gradient $= -24\sqrt{3}$

Question 7

Worked Solution

$x = 4t + 3, \ \ y = 4t + 8 + \frac{5}{2t}, \quad t \neq 0.$

Part (a): Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = 2$ as a simplified fraction.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4 - \frac{5}{2t^2}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 - 5/(2t^2)}{4} = 1 - \frac{5}{8t^2}$$

At $t = 2$: $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{5}{32} = \frac{27}{32}$.

(a) $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=2} = \frac{27}{32}$

Part (b): Show the Cartesian equation is $y = \frac{x^2 + ax + b}{x - 3}$, $x \neq 3$, and find a, b.

From $x = 4t + 3$: $t = \frac{x-3}{4}$.

$$y = 4 \cdot \frac{x-3}{4} + 8 + \frac{5}{2 \cdot (x-3)/4} = (x-3) + 8 + \frac{10}{x-3} = x + 5 + \frac{10}{x-3}$$
$$= \frac{(x+5)(x-3) + 10}{x-3} = \frac{x^2 + 2x - 15 + 10}{x-3} = \frac{x^2 + 2x - 5}{x-3}$$

(b) $y = \frac{x^2 + 2x - 5}{x - 3}$, so $a = 2$, $b = -5$.

End of Worked Solutions