

Question 1 (Jun 2005, Q9i [Modified])

Worked Solution

The function f is defined by $f(x) = \sqrt{mx + 7} - 4$, where $x \geq -\frac{7}{m}$ and m is a positive constant.

A sequence of transformations maps $y = \sqrt{x}$ to $y = f(x)$. We identify these transformations step by step.

Starting from $y = \sqrt{x}$, we rewrite $f(x)$ by factoring inside the square root:

$$f(x) = \sqrt{m\left(x + \frac{7}{m}\right)} - 4$$

Step 1 – Translation in the x -direction:

Replace x with $x + \frac{7}{m}$:

$$y = \sqrt{x} \longrightarrow y = \sqrt{x + \frac{7}{m}}$$

This is a **translation by $\frac{7}{m}$ units in the negative x -direction**, i.e. vector $\begin{pmatrix} -7/m \\ 0 \end{pmatrix}$.

Step 2 – Stretch in the x -direction:

Replace x with mx (equivalently, scale factor $\frac{1}{m}$ in the x -direction):

$$y = \sqrt{x + \frac{7}{m}} \longrightarrow y = \sqrt{mx + 7} = \sqrt{m\left(x + \frac{7}{m}\right)}$$

This is a **stretch parallel to the x -axis with scale factor $\frac{1}{m}$** .

Step 3 – Translation in the y -direction:

Subtract 4:

$$y = \sqrt{mx + 7} \longrightarrow y = \sqrt{mx + 7} - 4$$

This is a **translation by 4 units in the negative y -direction**, i.e. vector $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

The transformations are, in order:

1. Translation by $\frac{7}{m}$ units in the negative x -direction.
2. Stretch parallel to the x -axis with scale factor $\frac{1}{m}$.
3. Translation by 4 units in the negative y -direction.

(The two translations may be combined as a single step.)

Question 2 (Jun 2009, Q8i)**Worked Solution**

Give details of the pair of transformations which transforms $y = \ln x$ to $y = 2 \ln(x - 6)$.

We decompose the mapping step by step.

Step 1 – Translation in the x -direction:

Replace x with $x - 6$:

$$y = \ln x \longrightarrow y = \ln(x - 6)$$

This is a **translation by 6 units in the positive x -direction**, vector $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

Step 2 – Stretch in the y -direction:

Multiply by 2:

$$y = \ln(x - 6) \longrightarrow y = 2 \ln(x - 6)$$

This is a **stretch parallel to the y -axis with scale factor 2**.

1. Translation by 6 units in the positive x -direction.
2. Stretch parallel to the y -axis with scale factor 2.

(These may be applied in either order.)

Question 3 (Jan 2007, Q7i,ii)

Worked Solution

The curve $y = \ln x$ is transformed to $y = \ln\left(\frac{1}{2}x - a\right)$ by a translation followed by a stretch. Here $a > 0$.

Part (i) – Details of the translation and stretch:

Rewrite the target:

$$y = \ln\left(\frac{1}{2}x - a\right) = \ln\left(\frac{1}{2}(x - 2a)\right)$$

Translation: Replace x with $x - 2a$:

$$y = \ln x \longrightarrow y = \ln(x - 2a)$$

This is a **translation by $2a$ units in the positive x -direction.**

Stretch: Replace x with $\frac{1}{2}x$:

$$y = \ln(x - 2a) \longrightarrow y = \ln\left(\frac{1}{2}x - a\right)$$

This is a **stretch parallel to the x -axis with scale factor 2.**

(i) Translation by $2a$ in the positive x -direction; then stretch parallel to the x -axis, scale factor 2.

Part (ii) – Sketch of $y = \ln\left(\frac{1}{2}x - a\right)$:

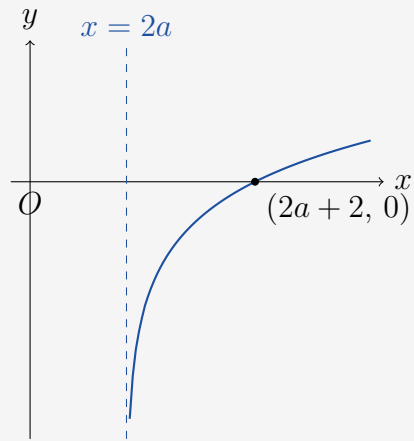
The curve $y = \ln x$ has a vertical asymptote at $x = 0$ and passes through $(1, 0)$.

After the translation, the asymptote moves to $x = 2a$ and the zero-crossing moves to $(2a + 1, 0)$.

After the stretch (scale factor 2 in x), the asymptote remains at $x = 2a$ (since the stretch is about $x = 0$, but note: the stretch acts on the translated curve, so it maps $(2a + 1, 0)$ to $(4a + 2, 0)$ and the asymptote moves to $x = 2a$).

The curve $y = \ln\left(\frac{1}{2}x - a\right)$:

- Has a vertical asymptote at $x = 2a$.
- Crosses the x -axis at $(2a + 2, 0)$ (since $\frac{1}{2}(2a + 2) - a = 1$ and $\ln 1 = 0$).
- Is an increasing function, passing through the first and fourth quadrants only (for $x > 2a$).



(ii) The sketch shows an increasing logarithmic curve with vertical asymptote $x = 2a$, crossing the x -axis at $(2a + 2, 0)$, existing only for $x > 2a$.

Question 4 (Jan 2008, Q6i)**Worked Solution**

Give details of the pair of geometrical transformations which transforms the graph of $y = -\sin^{-1}(x - 1)$ to the graph of $y = \sin^{-1} x$.

We work *from* $y = -\sin^{-1}(x - 1)$ *to* $y = \sin^{-1} x$.

Method – working backwards:

Starting from $y = -\sin^{-1}(x - 1)$:

Step 1 – Reflection in the x -axis:

Multiply y by -1 :

$$y = -\sin^{-1}(x - 1) \longrightarrow y = \sin^{-1}(x - 1)$$

This is a **reflection in the x -axis**.

Step 2 – Translation in the x -direction:

Replace $x - 1$ with x (i.e. translate by 1 in the positive x -direction):

$$y = \sin^{-1}(x - 1) \longrightarrow y = \sin^{-1} x$$

This is a **translation by 1 unit in the positive x -direction**.

1. Reflection in the x -axis.
2. Translation by 1 unit in the positive x -direction.

(Alternatively: reflection in the y -axis followed by translation 1 unit in the negative x -direction, applying in either order.)

Question 5 (Jan 2010, Q8i)

Worked Solution

The curve $y = \sqrt{x}$ can be transformed to $y = \sqrt{2x+3}$ by a stretch parallel to the y -axis followed by a translation.

We rewrite $y = \sqrt{2x+3}$:

$$y = \sqrt{2x+3} = \sqrt{2} \cdot \sqrt{x + \frac{3}{2}}$$

Step 1 – Stretch parallel to the y -axis:

The factor $\sqrt{2}$ outside gives a **stretch parallel to the y -axis with scale factor $\sqrt{2}$** .

After this stretch: $y = \sqrt{x} \longrightarrow y = \sqrt{2} \cdot \sqrt{x} = \sqrt{2x}$.

Step 2 – Translation in the x -direction:

Replace x with $x + \frac{3}{2}$:

$$y = \sqrt{2x} \longrightarrow y = \sqrt{2(x + \frac{3}{2})} = \sqrt{2x+3}$$

This is a **translation by $\frac{3}{2}$ units in the negative x -direction**, vector $\begin{pmatrix} -3/2 \\ 0 \end{pmatrix}$.

Scale factor of the stretch (parallel to y -axis): $\sqrt{2}$

Translation: $\frac{3}{2}$ units in the negative x -direction.

Question 6 (Jun 2010, Q2)

Worked Solution

The transformations R, S, T are defined as:

- R: reflection in the x -axis
- S: stretch in the x -direction with scale factor 3
- T: translation in the positive x -direction by 4 units

Part (i): The curve $y = \ln x$ is transformed by R then T. Find the resulting equation.

Apply R: Reflect $y = \ln x$ in the x -axis:

$$y = \ln x \longrightarrow y = -\ln x$$

Apply T: Translate by 4 units in positive x -direction (replace x with $x - 4$):

$$y = -\ln x \longrightarrow y = -\ln(x - 4)$$

(i) $y = -\ln(x - 4)$

Part (ii): Find, in terms of S and T, a sequence of transformations that maps $y = x^3$ to $y = \left(\frac{1}{9}x - 4\right)^3$.

We rewrite the target:

$$y = \left(\frac{1}{9}x - 4\right)^3 = \left(\frac{1}{9}(x - 36)\right)^3$$

Option: Apply T (translate +4 in x) then S (stretch factor 3 in x) then S again:

Actually, let's be systematic. We need $x \mapsto \frac{1}{9}x - 4$.

Applying S (scale factor 3 in x): $x \mapsto \frac{x}{3}$, so $y = x^3 \rightarrow y = \left(\frac{x}{3}\right)^3$.

Applying S again: $x \mapsto \frac{x}{9}$, giving $y = \left(\frac{x}{9}\right)^3$.

Applying T (replace x with $x - 4$): $y = \left(\frac{x-4}{9}\right)^3$. This is not quite right.

Let's reconsider. We want $y = \left(\frac{x}{9} - 4\right)^3$.

Apply S twice first ($x \rightarrow \frac{x}{9}$), then T ($x \rightarrow x - 36$):

$$y = x^3 \xrightarrow{S} y = \left(\frac{x}{3}\right)^3 \xrightarrow{S} y = \left(\frac{x}{9}\right)^3 \xrightarrow{\text{T by 36}} y = \left(\frac{x-36}{9}\right)^3 = \left(\frac{x}{9} - 4\right)^3$$

But T is defined as translation by exactly 4 units. So we need: apply T, then S, then S (or S^2):

Apply T: $y = (x - 4)^3$.

Apply S: $x \rightarrow \frac{x}{3}$, so $y = \left(\frac{x}{3} - 4\right)^3$.

Apply S: $x \rightarrow \frac{x}{3}$, so $y = \left(\frac{x}{9} - 4\right)^3$. ✓

(ii) Apply T, then S, then S (i.e. T followed by S twice, written T then S then S or T, S²).

Question 7 (Jun 2011, Q2)

Worked Solution

The curve $y = \ln x$ is transformed by:

- a reflection in the x -axis,
- followed by a stretch with scale factor 3 parallel to the y -axis,
- followed by a translation in the positive y -direction by $\ln 4$.

Find the equation of the resulting curve in the form $y = \ln(f(x))$.

Step 1 – Reflect in x -axis:

$$y = \ln x \longrightarrow y = -\ln x$$

Step 2 – Stretch scale factor 3 parallel to y -axis:

$$y = -\ln x \longrightarrow y = -3 \ln x$$

Step 3 – Translate $\ln 4$ in positive y -direction:

$$y = -3 \ln x \longrightarrow y = -3 \ln x + \ln 4$$

Now use logarithm laws to write in the form $y = \ln(f(x))$:

$$y = \ln 4 - 3 \ln x = \ln 4 - \ln x^3 = \ln \frac{4}{x^3} = \ln(4x^{-3})$$

$$y = \ln(4x^{-3})$$

Question 8 (Jan 2012, Q9)

Worked Solution

The function $f(x) = k(x^2 + 4x)$ where $k > 0$. The curve $y = x^2$ is transformed to $y = f(x)$ by: a translation parallel to the x -axis, then a translation parallel to the y -axis, then a stretch. Find details of these transformations.

First complete the square:

$$f(x) = k(x^2 + 4x) = k((x + 2)^2 - 4)$$

Step 1 – Translation parallel to the x -axis:

Replace x with $x + 2$ (translate 2 units in the negative x -direction):

$$y = x^2 \longrightarrow y = (x + 2)^2$$

Translation by 2 units in the negative x -direction.

Step 2 – Translation parallel to the y -axis:

Subtract 4:

$$y = (x + 2)^2 \longrightarrow y = (x + 2)^2 - 4$$

Translation by 4 units in the negative y -direction.

Step 3 – Stretch parallel to the y -axis:

Multiply by k :

$$y = (x + 2)^2 - 4 \longrightarrow y = k((x + 2)^2 - 4) = k(x^2 + 4x)$$

Stretch parallel to the y -axis with scale factor k .

1. Translation by 2 units in the negative x -direction.
2. Translation by 4 units in the negative y -direction.
3. Stretch parallel to the y -axis with scale factor k .

Question 9 (Jun 2006, Q8i,ii) [Requires knowledge of $R \cos(\theta - \alpha)$]

Worked Solution

Part (i): Express $5 \cos x + 12 \sin x$ in the form $R \cos(x - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$.

Expand $R \cos(x - \alpha) = R \cos \alpha \cos x + R \sin \alpha \sin x$.

Comparing coefficients:

$$R \cos \alpha = 5, \quad R \sin \alpha = 12$$

$$R = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\tan \alpha = \frac{12}{5} \implies \alpha = \arctan\left(\frac{12}{5}\right) \approx 67.4^\circ$$

$$(i) \quad 5 \cos x + 12 \sin x = 13 \cos(x - 67.4^\circ)$$

Part (ii): Give details of a pair of transformations which transforms $y = \cos x$ to $y = 5 \cos x + 12 \sin x$.

From part (i): $y = 5 \cos x + 12 \sin x = 13 \cos(x - 67.4^\circ)$.

Step 1 – Translation in the positive x -direction:

$$y = \cos x \longrightarrow y = \cos(x - 67.4^\circ)$$

Translation by 67.4° in the positive x -direction.

Step 2 – Stretch parallel to the y -axis:

$$y = \cos(x - 67.4^\circ) \longrightarrow y = 13 \cos(x - 67.4^\circ)$$

Stretch parallel to the y -axis with scale factor 13.

(ii)

1. Translation by 67.4° in the positive x -direction.
2. Stretch parallel to the y -axis with scale factor 13.

(These may be applied in either order.)

Question 10 (Jan 2009, Q7) [Requires knowledge of the modulus function]

Worked Solution

The curve $y = e^{kx} - a$, where k and a are constants.

Part (i): Give details of the pair of transformations which transforms $y = e^x$ to $y = e^{kx} - a$.

Step 1 – Stretch in the x -direction:

Replace x with kx (i.e. scale factor $\frac{1}{k}$ in the x -direction):

$$y = e^x \longrightarrow y = e^{kx}$$

Stretch parallel to the x -axis with scale factor $\frac{1}{k}$.

Step 2 – Translation in the y -direction:

$$y = e^{kx} \longrightarrow y = e^{kx} - a$$

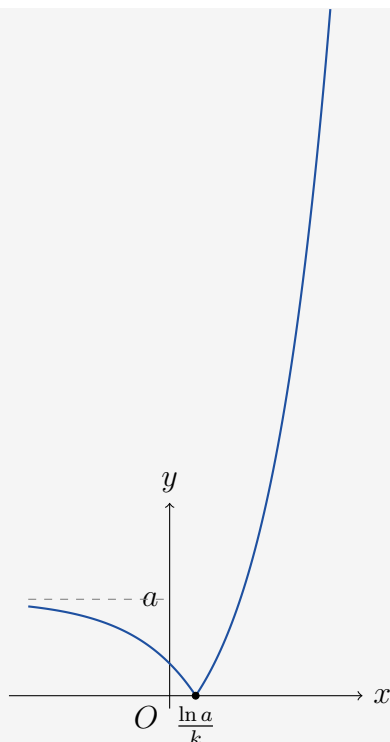
Translation by a units in the negative y -direction.

(i) Stretch parallel to the x -axis, scale factor $\frac{1}{k}$; then translation by a in the negative y -direction. (Either order.)

Part (ii): Sketch $y = |e^{kx} - a|$.

The curve $y = e^{kx} - a$ crosses the x -axis where $e^{kx} = a$, i.e. $x = \frac{\ln a}{k}$.

For $x < \frac{\ln a}{k}$: $e^{kx} < a$, so $e^{kx} - a < 0$; the modulus reflects this part in the x -axis.



(ii) The sketch shows the exponential curve reflected in the x -axis for $x < \frac{\ln a}{k}$, with a minimum touching the x -axis at $x = \frac{\ln a}{k}$, and y -intercept at $|1 - a|$.

Part (iii): Given $y = |e^{kx} - a|$ passes through $(0, 13)$ and $(\ln 3, 13)$, find k and a .

At $x = 0$: $|e^0 - a| = |1 - a| = 13$, so $a = 14$ or $a = -12$.

Since a must be positive (curve $y = e^{kx} - a$ with diagram showing it dips below x -axis):

$$a = 14$$

At $x = \ln 3$: $|e^{k \ln 3} - 14| = 13$, so $|3^k - 14| = 13$.

$$3^k - 14 = 13 \implies 3^k = 27 \implies k = 3$$

(The other case $3^k = 1$ gives $k = 0$, rejected.)

(iii) $a = 14$, $k = 3$.

End of Worked Solutions