Question 1

Worked Solution

(a) Express $3\sin\theta - 4\cos\theta$ in the form $R\sin(\theta-\alpha)$, $R>0$, $0<\alpha<90$.

Expand $R\sin(\theta-\alpha) = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$.

Matching: $R\cos\alpha = 3$, $R\sin\alpha = 4$.

$$R = \sqrt{9+16} = 5, \qquad \tan\alpha = \tfrac{4}{3} \Rightarrow \alpha = 53.13.$$

$3\sin\theta - 4\cos\theta = 5\sin(\theta - 53.13)$

(b)(i) Maximum temperature in the greenhouse ($G = 17 + 3\sin(15t) - 4\cos(15t)$).

$G = 17 + 5\sin(15t - 53.13)$. Maximum when sin = 1: $G_{\max} = 17 + 5 = 22$C.

Maximum temperature = 22C

(b)(ii) Time after midday when $G = 20$C.

$5\sin(15t - 53.13) = 3 \Rightarrow \sin(15t - 53.13) = 0.6$.

After-midday solution: $15t - 53.13 = 180 - 36.87 = 143.13$, so $15t = 196.26$, $t = 13.084$ h after 5 a.m. = 18:05 (6:05 p.m.).

6:05 p.m.

Question 2

Worked Solution

$D = 5 + 2\sin(30t)$, $0 \le t < 24$.

(a) Depth at 6:30 a.m. ($t = 6.5$).

$$D = 5 + 2\sin(195) = 5 + 2(-0.2588) \approx 4.48 \text{ m}.$$

$D \approx 4.48$ m

(b) Earliest time the boat (loaded by 8:30 a.m.) can leave when $D \ge 3.8$ m.

$\sin(30t) = -0.6 \Rightarrow 30t = 216.87$ or 323.13, giving $t = 7.23$ or $t = 10.77$.

$D < 3.8$ between $t \approx 7.23$ and $t \approx 10.77$. The boat is ready at $t = 8.5$, which falls in this interval, so it must wait until $t = 10.77$ h.

10.77 h $= 10$ h 46 min after midnight $=$ 10:46 a.m.

10:46 a.m. (or 10:47 a.m.)

Question 3

Worked Solution

$$H = \frac{140}{A + 45\sin 2t - 28\cos 2t}, \ 0 \leq t \leq T.$$

(a) Given nest is 20 m above sea level, find the equation for H.

At $t = 0$, $H = 20$: $20 = \dfrac{140}{A - 28} \Rightarrow A = 35$.

$$H = \frac{140}{35 + 45\sin 2t - 28\cos 2t}$$

(b) $45\sin 2t - 28\cos 2t \equiv 53\sin(2t - \alpha)$. **Find α.**

$\tan\alpha = \dfrac{28}{45} \Rightarrow \alpha = 31.9$.

$\alpha = 31.9$

(c) Minimum height of the seabird.

$$H_{\text{min}} = \frac{140}{35 + 53} = \frac{140}{88} \approx 1.59 \text{ m} = 159 \text{ cm}.$$

Minimum height ≈ 159 cm

(d) Limitation on T.

The denominator must remain positive (H must be defined and positive). The denominator first reaches zero at $T \approx 126.6$ s.

Model valid for $0 < T < 126.6$ s

Question 4

Worked Solution

(a) Express $1.5\sin\theta - 1.2\cos\theta$ in the form $R\sin(\theta-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$.

$R = \sqrt{1.5^2 + 1.2^2} = \sqrt{3.69} \approx 1.921.$

$$\tan\alpha = \frac{1.2}{1.5} = 0.8 \Rightarrow \alpha = 0.675 \text{ rad.}$$

$1.5\sin\theta - 1.2\cos\theta \approx 1.921\sin(\theta - 0.675)$

(b) $H = 3 + 1.5\sin\left(\frac{\pi t}{6}\right) - 1.2\cos\left(\frac{\pi t}{6}\right)$. Minimum H and when it occurs.

$H = 3 + 1.921\sin\left(\frac{\pi t}{6} - 0.675\right).$

Minimum when sin $= -1$: $H_{\text{min}} = 3 - 1.921 \approx 1.08$ m.

$\frac{\pi t}{6} - 0.675 = \frac{3\pi}{2} \Rightarrow t = \frac{6}{\pi}\left(\frac{3\pi}{2} + 0.675\right) \approx 10.29$ h.

$H_{\text{min}} \approx 1.08$ m at $t \approx 10.29$ h after midday

(c) Times when $H = 4$ m (to nearest minute).

$\sin\left(\frac{\pi t}{6} - 0.675\right) = \frac{1}{1.921} \approx 0.5206.$

Solution 1: $\frac{\pi t}{6} - 0.675 = 0.548 \Rightarrow t \approx 2.33$ h $\approx$ 2:20 p.m.

Solution 2: $\frac{\pi t}{6} - 0.675 = \pi - 0.548 = 2.594 \Rightarrow t \approx 6.24$ h $\approx$ 6:14 p.m.

Approximately 2:20 p.m. and 6:15 p.m.

Question 5

Worked Solution

(a) Express $2\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$.

$R = \sqrt{4+16} = 2\sqrt{5}$, $\tan\alpha = 2$, $\alpha \approx 1.11$ rad.

$$2\sin x - 4\cos x = 2\sqrt{5}\sin(x - 1.11)$$

(b) Maximum and minimum hours of daylight: $H = 12 + 2\cdot\left(2\sin\frac{2\pi t}{365} - 4\cos\frac{2\pi t}{365}\right) = 12 + 4\sqrt{5}\sin\left(\frac{2\pi t}{365} - 1.11\right)$.

Maximum: $12 + 4\sqrt{5} \approx 20.9$ h. Minimum: $12 - 4\sqrt{5} \approx 3.06$ h.

$$\text{Maximum} \approx 20.9\text{ h}; \quad \text{Minimum} \approx 3.06\text{ h}$$

(c) Values of t when $H = 17$.

$4\sqrt{5}\sin\left(\frac{2\pi t}{365} - 1.11\right) = 5 \Rightarrow \sin(\ldots) = \dfrac{\sqrt{5}}{4} \approx 0.559.$

$t \approx 99$ days and $t \approx 212$ or 213 days.

$$t \approx 99 \text{ days and } t \approx 212 \text{ or } 213 \text{ days}$$

Question 6

Worked Solution

(a) Write $2\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$.

$R = \sqrt{5}$, $\tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26.6$.

$$2\sin\theta - \cos\theta = \sqrt{5}\sin(\theta - 26.6)$$

(b) Sketch $y = |2\sin\theta - \cos\theta|$ for $0 \leq \theta < 360$.

The function $\sqrt{5}\sin(\theta - 26.6)$ has zeros at $\theta = 26.6$ and 206.6. Taking the modulus reflects the negative arch upward.

Key points on sketch:

- (0, 1) [since $|-1| = 1$]
- (26.6, 0) [zero crossing]
- (116.6, $\sqrt{5}$) [maximum]
- (206.6, 0) [zero crossing, becomes reflected cusp]
- (296.6, $\sqrt{5}$) [second maximum after reflection]

(c)(i) Maximum value of $f(t) = 5 + |2\sin(15t) - \cos(15t)|$.

$f(t) = 5 + \sqrt{5}|\sin(15t - 26.6)|$. Maximum $= 5 + \sqrt{5}$.

$$f_{\max} = 5 + \sqrt{5}$$

(c)(ii) Largest $t \in [0, 24)$ at which the maximum occurs.

$|\sin(15t - 26.6)| = 1 \Rightarrow 15t - 26.6 = 270$ gives $t = 19.77 \approx 19.8$.

Next would be $15t - 26.6 = 90 + 360 = 450 \Rightarrow t = 31.8$ (out of range).

Largest $t \approx 19.8$ (to 1 d.p.)

End of Worked Solutions