Question 1 (Jun 2006, Q6)

Worked Solution

(i)(a) John pays £100 in month 1, increasing by £5 each month. This is an **arithmetic progression** with $a = 100$, $d = 5$.

Final month (month 240):

$$u_{240} = a + (n-1)d = 100 + 239 \times 5 = 100 + 1195 = £1295.$$

$u_{240} = £1295$

(i)(b) Total over 240 months:

$$S_{240} = \tfrac{n}{2}(a + l) = \tfrac{240}{2}(100 + 1295) = 120 \times 1395 = £167\,400.$$

$S_{240} = £167\,400$

(ii) Rachel's payments form a **geometric progression** with $a = 100$, $u_{240} = 1500$.

Find r: $u_{240} = ar^{239} \Rightarrow 100r^{239} = 1500 \Rightarrow r^{239} = 15 \Rightarrow r = 15^{1/239} \approx 1.01139$.

Total: $S_{240} = \dfrac{a(r^{240} - 1)}{r - 1} = \dfrac{100(1.01139^{240} - 1)}{0.01139} \approx £124\,359$.

Rachel pays approximately £124 359 in total (3 s.f. or better).

Question 2 (Jun 2007, Q7)

Worked Solution

(a) Arithmetic progression: first term 12, sum of first 70 terms is 12 915. Find d.

Use $S_n = \frac{n}{2}[2a + (n-1)d]$:

$$S_{70} = \tfrac{70}{2}[2(12) + 69d] = 35(24 + 69d) = 12\,915.$$

$$24 + 69d = 369 \Rightarrow 69d = 345 \Rightarrow d = 5.$$

$d = 5$

(b) Geometric progression: second term $= -4$, sum to infinity $= 9$. Find r.

$ar = -4$ and $\dfrac{a}{1-r} = 9$.

From the first: $a = \dfrac{-4}{r}$. Substitute into the second:

$$\frac{-4/r}{1-r} = 9 \Rightarrow -4 = 9r(1-r) \Rightarrow 9r^2 - 9r - 4 = 0.$$

Factorise: $(3r-4)(3r+1) = 0 \Rightarrow r = \frac{4}{3}$ or $r = -\frac{1}{3}$.
For the GP to converge (sum to infinity exists), we need $|r| < 1$, so $r = \frac{4}{3}$ is rejected.

$r = -\dfrac{1}{3}$

Question 3 (Jun 2008, Q10)

Worked Solution

Jamie trains daily. Swimming: 2 km constant. Running: AP with $a = 2$, $d = 0.5$. Cycling: GP with $a = 2$, $r = 1.1$.

(i) Distance Jamie runs on Day 15.

$$u_{15} = 2 + 14 \times 0.5 = 2 + 7 = 9 \text{ km}.$$

9 km

(ii) Verify that the distance cycled first exceeds 12 km on Day 20.

$u_n = 2 \times (1.1)^{n-1}$. Check Day 20: $u_{20} = 2 \times (1.1)^{19}$.

$(1.1)^{19} \approx 6.116$, so $u_{20} \approx 12.2 > 12$. ✓

Check Day 19: $u_{19} = 2 \times (1.1)^{18} \approx 2 \times 5.560 = 11.1 < 12$.

So Day 20 is the first day it exceeds 12 km. ✓

(iii) Find the day on which total distance cycled first exceeds 200 km.

$S_n = \dfrac{2(1.1^n - 1)}{0.1} = 20(1.1^n - 1) > 200.$

$1.1^n - 1 > 10 \Rightarrow 1.1^n > 11.$

Taking logs: $n \ln 1.1 > \ln 11 \Rightarrow n > \dfrac{\ln 11}{\ln 1.1} = \dfrac{2.3979}{0.09531} \approx 25.16.$

So $n = 26$ (Day 26).

Day 26

(iv) Total distance (swim + run + cycle) up to and including Day 30.

Swimming (constant 2 km/day): $30 \times 2 = 60$ km.

Running (AP, $a = 2$, $d = 0.5$, $n = 30$): $S_{30} = \frac{30}{2}(2 \times 2 + 29 \times 0.5) = 15(4 + 14.5) = 15 \times 18.5 = 277.5$ km.

Cycling (GP, $a = 2$, $r = 1.1$, $n = 30$): $S_{30} = \dfrac{2(1.1^{30} - 1)}{0.1} = 20(1.1^{30} - 1).$

$1.1^{30} \approx 17.449$, so cycling $\approx 20 \times 16.449 = 329.0$ km.

Total $= 60 + 277.5 + 329.0 \approx 666$ km.

Total ≈ 666 km

Question 4 (Jun 2010, Q4)

Worked Solution

The sequence $u_1, u_2, u_3, \dots$ is defined by $u_n = 5n + 1$.

(i) State the values of u_1, u_2, u_3.

$$u_1 = 6, \quad u_2 = 11, \quad u_3 = 16.$$

$u_1 = 6, \quad u_2 = 11, \quad u_3 = 16$

(ii) Evaluate $\sum_{n=1}^{40} u_n$.

This is an AP with $a = 6$, $d = 5$, $n = 40$.

$$S_{40} = \tfrac{40}{2}[2(6) + 39(5)] = 20[12 + 195] = 20 \times 207 = 4140.$$

4140

(iii) Another sequence $w_1 = 2$, $w_{n+1} = 5w_n + 1$. Find p such that $u_p = w_3$.

$w_2 = 5(2) + 1 = 11$, $w_3 = 5(11) + 1 = 56$.

$5p + 1 = 56 \Rightarrow 5p = 55 \Rightarrow p = 11$.

$p = 11$

Question 5 (Jun 2010, Q9)

Worked Solution

GP with first term a, common ratio r (all terms different). The 1st, 2nd, 4th terms of the GP form the first three terms of an AP.

(i) Show that $r^3 - 2r + 1 = 0$.

The AP terms are $a,\ ar,\ ar^3$. For an AP, the common difference is constant:

$$ar - a = ar^3 - ar.$$

$$a(r-1) = ar(r^2-1) = ar(r-1)(r+1).$$

Since the terms are all different, $a \neq 0$ and $r \neq 1$, so divide by $a(r-1)$:

$$1 = r(r+1) \Rightarrow r^2 + r - 1 = 0 \ldots \text{wait, let us redo more carefully.}$$

AP condition: $2(ar) = a + ar^3 \Rightarrow 2ar = a + ar^3$.

Divide by a: $2r = 1 + r^3 \Rightarrow r^3 - 2r + 1 = 0$. ✓

$r^3 - 2r + 1 = 0$ (shown)

(ii) Given the GP converges, find the exact value of r.

Factor: $r^3 - 2r + 1 = (r-1)(r^2 + r - 1) = 0$.

$r = 1$ gives identical terms (rejected). For $r^2 + r - 1 = 0$:

$$r = \frac{-1 \pm \sqrt{5}}{2}.$$

For convergence, $|r| < 1$: $r = \dfrac{-1+\sqrt{5}}{2} \approx 0.618$ or $r = \dfrac{-1-\sqrt{5}}{2} \approx -1.618$ (rejected).

$r = \dfrac{-1+\sqrt{5}}{2}$

(iii) Given $S_\infty = 3 + \sqrt{5}$, find a.

$$S_\infty = \frac{a}{1-r} = 3 + \sqrt{5}, \quad \text{where } 1 - r = 1 - \tfrac{-1+\sqrt{5}}{2} = \tfrac{3-\sqrt{5}}{2}.$$

$$a = (3+\sqrt{5}) \cdot \tfrac{3-\sqrt{5}}{2} = \tfrac{9-5}{2} = \tfrac{4}{2} = 2.$$

$a = 2$

Question 6 (Jan 2012, Q6)

Worked Solution

Sequence defined by $u_n = 85 - 5n$ for $n \geq 1$.

(i) Write down u_1, u_2, u_3.

$$u_1 = 80, \quad u_2 = 75, \quad u_3 = 70.$$

$u_1 = 80, \quad u_2 = 75, \quad u_3 = 70$

(ii) Find $\sum_{n=1}^{20} u_n$.

AP with $a = 80$, $d = -5$, $n = 20$:

$$S_{20} = \tfrac{20}{2}[2(80) + 19(-5)] = 10[160 - 95] = 10 \times 65 = 650.$$

650

(iii) u_1, u_5, u_p form the first three terms of a GP. Find p.

$u_1 = 80$, $u_5 = 60$. Common ratio: $r = \dfrac{60}{80} = \dfrac{3}{4}$.

Third GP term: $80 \times \left(\tfrac{3}{4}\right)^2 = 80 \times \tfrac{9}{16} = 45$.

$u_p = 85 - 5p = 45 \Rightarrow 5p = 40 \Rightarrow p = 8$.

$p = 8$

(iv) Find S_∞ of the GP in part (iii).

$$S_\infty = \frac{80}{1 - 3/4} = \frac{80}{1/4} = 320.$$

$S_\infty = 320$

Question 7 (Jan 2013, Q6)

Worked Solution

(i) The first three terms of an AP are $2x$, $x+4$, $2x-7$. Find x.

For an AP the common difference is constant:

$$(x+4) - 2x = (2x-7) - (x+4).$$

$$4 - x = x - 11 \Rightarrow 2x = 15 \Rightarrow x = 7.5.$$

$x = 7.5$

(ii)(a) Verify that when $x = 8$ the terms form a GP, and find S_∞.

With $x = 8$: terms are $16,\ 12,\ 9$.

Check ratios: $\dfrac{12}{16} = \dfrac{3}{4}$ and $\dfrac{9}{12} = \dfrac{3}{4}$. Common ratio confirmed. ✓

$$S_\infty = \frac{16}{1 - 3/4} = \frac{16}{1/4} = 64.$$

$S_\infty = 64$

(ii)(b) Find the other value of x that gives a GP.

For a GP: $\dfrac{x+4}{2x} = \dfrac{2x-7}{x+4}$, so $(x+4)^2 = 2x(2x-7)$:

$$x^2 + 8x + 16 = 4x^2 - 14x \Rightarrow 3x^2 - 22x - 16 = 0.$$

Factorise: $(3x+2)(x-8) = 0 \Rightarrow x = 8$ or $x = -\frac{2}{3}$.

The other value is $x = -\dfrac{2}{3}$

Question 8 (Jun 2013, Q8)

Worked Solution

Sarah uses 6 g in experiment 1, 7.8 g in experiment 2.

(i) AP: find total used in first 30 experiments.

$a = 6$, $d = 7.8 - 6 = 1.8$, $n = 30$.

$$S_{30} = \tfrac{30}{2}[2(6) + 29(1.8)] = 15[12 + 52.2] = 15 \times 64.2 = 963 \text{ g}.$$

963 g

(ii) GP: show $1.3^N \leq 91$ and find N.

$a = 6$, $r = \dfrac{7.8}{6} = 1.3$. Total after N experiments:

$$S_N = \frac{6(1.3^N - 1)}{0.3} \leq 1800.$$

$$6(1.3^N - 1) \leq 540 \Rightarrow 1.3^N - 1 \leq 90 \Rightarrow 1.3^N \leq 91. \quad \checkmark$$

Taking logs:

$$N \ln 1.3 \leq \ln 91 \Rightarrow N \leq \frac{\ln 91}{\ln 1.3} = \frac{4.5109}{0.2624} \approx 17.19.$$

So the greatest integer value is $N = 17$.

$N = 17$

End of Worked Solutions