Question 1 (OCR 4724, Jun 2006, Q3)

Worked Solution

(i) Express $\dfrac{3-2x}{x(3-x)}$ in partial fractions.

Write $\dfrac{3-2x}{x(3-x)} = \dfrac{A}{x} + \dfrac{B}{3-x}$.

Multiply both sides by $x(3-x)$:

$$3 - 2x = A(3-x) + Bx.$$

Set $x = 0$: $3 = 3A \Rightarrow A = 1$.

Set $x = 3$: $3 - 6 = 3B \Rightarrow B = -1$.

$$\frac{3-2x}{x(3-x)} = \frac{1}{x} - \frac{1}{3-x}$$

(ii) Show that $\displaystyle\int_1^2 \frac{3-2x}{x(3-x)}\,dx = 0$.

Using the partial fractions from (i):

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{3-x}\right) dx = \Big[\ln x + \ln(3-x)\Big]_1^2.$$

Note: $\displaystyle\int \frac{-1}{3-x}\,dx = \ln(3-x)$ (since $\frac{d}{dx}(3-x) = -1$ cancels the minus sign).

At $x = 2$: $\ln 2 + \ln 1 = \ln 2$.

At $x = 1$: $\ln 1 + \ln 2 = \ln 2$.

$$\ln 2 - \ln 2 = 0. \quad \checkmark$$

$$\int_1^2 \frac{3-2x}{x(3-x)}\,dx = 0 \qquad \text{(shown)}$$

(iii) What does this indicate about the graph of $y = \dfrac{3-2x}{x(3-x)}$ between $x = 1$ and $x = 2$?

The result indicates that the graph crosses the x-axis between $x = 1$ and $x = 2$, so there are equal areas above and below the axis (the positive and negative areas cancel exactly).

Question 2 (OCR 4724, Jan 2007, Q6)

Worked Solution

(i) Express $\frac{2x+1}{(x-3)^2}$ in the form $\frac{A}{x-3}+\frac{B}{(x-3)^2}$.

Write $2x+1 = A(x-3)+B$.

Set $x=3$: $7 = B \Rightarrow B = 7$.

Comparing coefficients of x: $A = 2$.

$$\frac{2x+1}{(x-3)^2} = \frac{2}{x-3} + \frac{7}{(x-3)^2}$$

(ii) Hence find the exact value of $\int_4^{10} \frac{2x+1}{(x-3)^2}\,dx$, in the form $a + b\ln c$.

Integrate each term:

$$\int \frac{2}{x-3}\,dx = 2\ln|x-3|, \qquad \int \frac{7}{(x-3)^2}\,dx = -\frac{7}{x-3}.$$

Let $F(x) = 2\ln|x-3| - \frac{7}{x-3}$.

Apply limits:

$$F(10) = 2\ln 7 - \tfrac{7}{7} = 2\ln 7 - 1.$$
$$F(4) = 2\ln 1 - \tfrac{7}{1} = 0 - 7 = -7.$$
$$F(10) - F(4) = 2\ln 7 - 1 + 7 = 6 + 2\ln 7.$$

$6 + 2\ln 7$

Question 3 (OCR 4724, Jun 2009, Q6)

Worked Solution

Let $f(x) = \dfrac{4x}{(x-5)(x-3)^2}$.

(i) Express $f(x)$ in the form $\dfrac{A}{x-5} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}$.

Multiply through by $(x-5)(x-3)^2$:

$$4x = A(x-3)^2 + B(x-3)(x-5) + C(x-5).$$

Set $x = 5$: $20 = A(4) \Rightarrow A = 5$.

Set $x = 3$: $12 = C(-2) \Rightarrow C = -6$.

Compare coefficients of x^2: $0 = A + B \Rightarrow B = -5$.

Check: coefficient of x^0*:* $A(9) + B(-3)(-5) + C(-5) = 45 + (-5)(15) + (-6)(-5) = 45 - 75 + 30 = 0$. ✓

$$\frac{5}{x-5} - \frac{5}{x-3} - \frac{6}{(x-3)^2}$$

(ii) Hence find the exact value of $\displaystyle\int_1^2 f(x)\,dx$.

Integrate each term:

$$\int \frac{5}{x-5}\,dx = 5\ln|x-5|, \quad \int \frac{-5}{x-3}\,dx = -5\ln|x-3|, \quad \int \frac{-6}{(x-3)^2}\,dx = \frac{6}{x-3}.$$

Let $F(x) = 5\ln|x-5| - 5\ln|x-3| + \dfrac{6}{x-3}$.

Apply limits:

$$F(2) = 5\ln 3 - 5\ln 1 + \tfrac{6}{-1} = 5\ln 3 - 6.$$

$$F(1) = 5\ln 4 - 5\ln 2 + \tfrac{6}{-2} = 5\ln 2 - 3.$$

$$F(2) - F(1) = 5\ln 3 - 6 - 5\ln 2 + 3 = 5\ln\tfrac{3}{2} - 3.$$

Also note $5\ln\frac{3}{4} - 5\ln\frac{1}{2}$; using the mark scheme approach the exact answer is:

$$5\ln\frac{3}{2} - 3$$

Question 4 (OCR 4724, Jun 2007, Q7)

Worked Solution

(i) Find the quotient and remainder when $2x^3 + 3x^2 + 9x + 12$ is divided by $x^2 + 4$.

Perform polynomial long division:

$$2x^3 + 3x^2 + 9x + 12 \div (x^2 + 4).$$

$2x^3 \div x^2 = 2x$. Multiply: $2x(x^2 + 4) = 2x^3 + 8x$. Subtract: $3x^2 + x + 12$.

$3x^2 \div x^2 = 3$. Multiply: $3(x^2 + 4) = 3x^2 + 12$. Subtract: x.

Quotient $= 2x + 3$, Remainder $= x$

(ii) Hence express $\dfrac{2x^3 + 3x^2 + 9x + 12}{x^2 + 4}$ in the form $Ax + B + \dfrac{Cx + D}{x^2 + 4}$.

From (i): $\dfrac{2x^3 + 3x^2 + 9x + 12}{x^2 + 4} = 2x + 3 + \dfrac{x}{x^2 + 4}$.

$A = 2,\ B = 3,\ C = 1,\ D = 0$

(iii) Use the result of (ii) to find the exact value of $\displaystyle\int_1^3 \frac{2x^3 + 3x^2 + 9x + 12}{x^2 + 4}\,dx$.

Integrate each part:

$$\int (2x + 3)\,dx = x^2 + 3x, \qquad \int \frac{x}{x^2 + 4}\,dx = \tfrac{1}{2}\ln(x^2 + 4).$$

Note: $\frac{d}{dx}(x^2 + 4) = 2x$, so $\displaystyle\int \frac{x}{x^2 + 4}\,dx = \tfrac{1}{2}\ln(x^2 + 4)$.

Let $F(x) = x^2 + 3x + \tfrac{1}{2}\ln(x^2 + 4)$.

Apply limits:

$$F(3) = 9 + 9 + \tfrac{1}{2}\ln 13 = 18 + \tfrac{1}{2}\ln 13.$$
$$F(1) = 1 + 3 + \tfrac{1}{2}\ln 5 = 4 + \tfrac{1}{2}\ln 5.$$
$$F(3) - F(1) = 14 + \tfrac{1}{2}\ln 13 - \tfrac{1}{2}\ln 5 = 14 + \tfrac{1}{2}\ln \tfrac{13}{5}.$$

$$14 + \frac{1}{2}\ln\frac{13}{5}$$

End of Worked Solutions