Question 1 (OCR 4724, Jun 2006, Q8i)

Worked Solution

Show that $\int \cos^2 6x \, dx = \frac{1}{2}x + \frac{1}{24}\sin 12x + c$.

Use the double-angle identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ with $\theta = 6x$:

$$\cos^2 6x = \tfrac{1}{2}(1+\cos 12x).$$

Integrate term by term:

$$\int \cos^2 6x \, dx = \int \tfrac{1}{2}(1+\cos 12x)\, dx = \tfrac{1}{2}x + \tfrac{1}{2}\cdot\tfrac{1}{12}\sin 12x + c.$$

$\int \cos^2 6x \, dx = \frac{1}{2}x + \frac{1}{24}\sin 12x + c$ ✓

Question 2 (OCR 4724, Jun 2012, Q7)

Worked Solution

Find the exact value of $\int_0^{\frac{1}{8}\pi} (1 - \sin 3x)^2 \, dx$.

Step 1: Expand the square.

$$(1 - \sin 3x)^2 = 1 - 2\sin 3x + \sin^2 3x.$$

Step 2: Replace $\sin^2 3x$ **using** $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ with $\theta = 3x$:

$$\sin^2 3x = \tfrac{1}{2}(1 - \cos 6x).$$

So the integrand becomes:

$$1 - 2\sin 3x + \tfrac{1}{2}(1 - \cos 6x) = \tfrac{3}{2} - 2\sin 3x - \tfrac{1}{2}\cos 6x.$$

Step 3: Integrate.

$$\int_0^{\pi/8} \left(\tfrac{3}{2} - 2\sin 3x - \tfrac{1}{2}\cos 6x\right) dx = \left[\tfrac{3}{2}x + \tfrac{2}{3}\cos 3x - \tfrac{1}{12}\sin 6x\right]_0^{\pi/8}.$$

Step 4: Evaluate at the limits.

At $x = \dfrac{\pi}{8}$:

$$\tfrac{3}{2} \cdot \tfrac{\pi}{8} + \tfrac{2}{3}\cos\tfrac{3\pi}{8} - \tfrac{1}{12}\sin\tfrac{6\pi}{8} = \tfrac{3\pi}{16} + \tfrac{2}{3}\cos\tfrac{3\pi}{8} - \tfrac{1}{12}\sin\tfrac{3\pi}{4}.$$

Note $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{3\pi}{8} = \sin\frac{\pi}{8}$, but using the mark-scheme approach directly:

At $x = 0$: $0 + \frac{2}{3}(1) - 0 = \frac{2}{3}$.

Subtracting: the exact value simplifies to

$$\tfrac{3\pi}{16} - \tfrac{2}{3} + \tfrac{2}{3}\cos\tfrac{3\pi}{8} - \tfrac{1}{12} \cdot \tfrac{\sqrt{2}}{2}.$$

Following the mark-scheme route (verify $\frac{3}{2}x$ gives $\frac{3\pi}{16}$ and the $-\frac{2}{3}$ comes from subtracting the lower limit's $\frac{2}{3}$):

$$\frac{3\pi}{16} - \frac{2}{3} + \frac{2}{3}\cos\frac{3\pi}{8} - \frac{\sqrt{2}}{24}$$

Note: The mark scheme quotes $\frac{1}{4}\pi$ + their $-\frac{2}{3}$ for the boundary terms from the two integrations.

Question 3 (OCR 4724, Jun 2016, Q2)

Worked Solution

Find the exact value of $\int_{\frac{1}{16}\pi}^{\frac{1}{8}\pi} (9 - 6\cos^2 4x)\,dx$.

Step 1: Replace $\cos^2 4x$**.**

$$\cos^2 4x = \tfrac{1}{2}(1 + \cos 8x).$$

So:

$$9 - 6\cos^2 4x = 9 - 3(1 + \cos 8x) = 6 - 3\cos 8x.$$

Step 2: Integrate.

$$\int (6 - 3\cos 8x)\,dx = 6x - \tfrac{3}{8}\sin 8x + c.$$

Let $F(x) = 6x - \frac{3}{8}\sin 8x$.

Step 3: Apply limits.

$$F\left(\tfrac{\pi}{8}\right) = \tfrac{6\pi}{8} - \tfrac{3}{8}\sin\pi = \tfrac{3\pi}{4} - 0 = \tfrac{3\pi}{4}.$$

$$F\left(\tfrac{\pi}{16}\right) = \tfrac{6\pi}{16} - \tfrac{3}{8}\sin\tfrac{\pi}{2} = \tfrac{3\pi}{8} - \tfrac{3}{8}.$$

$$F\left(\tfrac{\pi}{8}\right) - F\left(\tfrac{\pi}{16}\right) = \tfrac{3\pi}{4} - \tfrac{3\pi}{8} + \tfrac{3}{8} = \tfrac{3\pi}{8} + \tfrac{3}{8}.$$

$$\frac{3\pi}{8} + \frac{3}{8} = \frac{3(\pi + 1)}{8}$$

Question 4 (OCR 4724, Jun 2013, Q5)

Worked Solution

(i) Show that $\dfrac{1}{1-\tan x} - \dfrac{1}{1+\tan x} \equiv \tan 2x$.

Combine over a common denominator $(1-\tan x)(1+\tan x) = 1 - \tan^2 x$:

$$\frac{(1+\tan x)-(1-\tan x)}{1-\tan^2 x} = \frac{2\tan x}{1-\tan^2 x} = \tan 2x. \quad \checkmark$$

(using the double-angle formula $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.)

(ii) Hence evaluate $\displaystyle\int_{\frac{1}{12}\pi}^{\frac{1}{6}\pi} \left(\frac{1}{1-\tan x} - \frac{1}{1+\tan x} \right) dx$, giving your answer in the form $a \ln b$.

By part (i) this equals $\displaystyle\int_{\pi/12}^{\pi/6} \tan 2x \, dx$.

Integrate: $\displaystyle\int \tan 2x \, dx = \tfrac{1}{2} \ln|\sec 2x| + c = -\tfrac{1}{2} \ln|\cos 2x| + c$.

Let $F(x) = -\frac{1}{2} \ln|\cos 2x|$.

Apply limits:

$$F\left(\tfrac{\pi}{6}\right) = -\tfrac{1}{2} \ln\left|\cos \tfrac{\pi}{3}\right| = -\tfrac{1}{2} \ln \tfrac{1}{2} = \tfrac{1}{2} \ln 2.$$

$$F\left(\tfrac{\pi}{12}\right) = -\tfrac{1}{2} \ln\left|\cos \tfrac{\pi}{6}\right| = -\tfrac{1}{2} \ln \tfrac{\sqrt{3}}{2} = \tfrac{1}{2} \ln \tfrac{2}{\sqrt{3}} = \tfrac{1}{2} \ln \tfrac{2}{\sqrt{3}}.$$

$$F\left(\tfrac{\pi}{6}\right) - F\left(\tfrac{\pi}{12}\right) = \tfrac{1}{2} \ln 2 - \tfrac{1}{2} \ln \tfrac{2}{\sqrt{3}} = \tfrac{1}{2} \ln\left(2 \cdot \tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{2} \ln \sqrt{3} = \tfrac{1}{4} \ln 3.$$

$\dfrac{1}{4} \ln 3$

Question 5 (OCR 4724, Jan 2010, Q3)

Worked Solution

By expressing $\cos 2x$ **in terms of** $\cos x$**, find the exact value of** $\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} \frac{\cos 2x}{\cos^2 x}\,dx$.

Step 1: Substitute $\cos 2x = 2\cos^2 x - 1$**.**

$$\frac{\cos 2x}{\cos^2 x} = \frac{2\cos^2 x - 1}{\cos^2 x} = 2 - \sec^2 x.$$

Step 2: Integrate.

$$\int (2 - \sec^2 x)\,dx = 2x - \tan x + c.$$

Let $F(x) = 2x - \tan x$.

Step 3: Apply limits.

$$F\left(\tfrac{\pi}{3}\right) = \tfrac{2\pi}{3} - \tan\tfrac{\pi}{3} = \tfrac{2\pi}{3} - \sqrt{3}.$$

$$F\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{2} - \tan\tfrac{\pi}{4} = \tfrac{\pi}{2} - 1.$$

$$F\left(\tfrac{\pi}{3}\right) - F\left(\tfrac{\pi}{4}\right) = \tfrac{2\pi}{3} - \sqrt{3} - \tfrac{\pi}{2} + 1 = \tfrac{\pi}{6} - \sqrt{3} + 1.$$

$$\frac{\pi}{6} - \sqrt{3} + 1$$

Question 6 (OCR 4724, Jan 2011, Q3)

Worked Solution

(i) Show that the derivative of $\sec x$ can be written as $\sec x \tan x$.

Write $\sec x = \dfrac{1}{\cos x} = (\cos x)^{-1}$.

Differentiate using the chain rule (or quotient rule):

$$\frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x. \quad \checkmark$$

(ii) Find $\displaystyle\int \frac{\tan x}{\sqrt{1+\cos 2x}}\, dx$.

Step 1: Simplify the denominator.

Use $\cos 2x = 2\cos^2 x - 1$, so $1 + \cos 2x = 2\cos^2 x$.

$$\sqrt{1+\cos 2x} = \sqrt{2\cos^2 x} = \sqrt{2}\,|\cos x| = \sqrt{2}\cos x \quad \text{(taking positive root)}.$$

Step 2: Rewrite the integrand.

$$\frac{\tan x}{\sqrt{2}\cos x} = \frac{\sin x/\cos x}{\sqrt{2}\cos x} = \frac{\sin x}{\sqrt{2}\cos^2 x} = \frac{1}{\sqrt{2}} \cdot \frac{\tan x}{\cos x} = \frac{1}{\sqrt{2}} \sec x \tan x.$$

Step 3: Integrate.

Since $\dfrac{d}{dx}(\sec x) = \sec x \tan x$:

$$\int \frac{1}{\sqrt{2}} \sec x \tan x \, dx = \frac{1}{\sqrt{2}} \sec x + c.$$

$$\frac{1}{\sqrt{2}} \sec x + c$$

Question 7 (OCR 4724, Jun 2014, Q4)

Worked Solution

Show that $\int_0^{\frac{1}{4}\pi} \frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x}\, dx = \frac{1}{2}\ln 2.$

Step 1: Rewrite numerator and denominator using double-angle formulae.

Numerator: $1 - 2\sin^2 x = \cos 2x$ (since $\cos 2x = 1 - 2\sin^2 x$).

Denominator: $1 + 2\sin x \cos x = 1 + \sin 2x$.

So the integral becomes:

$$\int_0^{\pi/4} \frac{\cos 2x}{1 + \sin 2x}\, dx.$$

Step 2: Integrate using the standard form $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$.

Here the numerator is $\cos 2x$ and the denominator is $1 + \sin 2x$. Note $\frac{d}{dx}(1 + \sin 2x) = 2\cos 2x$, so:

$$\int \frac{\cos 2x}{1 + \sin 2x}\, dx = \frac{1}{2}\ln|1 + \sin 2x| + c.$$

Let $F(x) = \frac{1}{2}\ln(1 + \sin 2x)$.

Step 3: Apply limits.

$$F\left(\tfrac{\pi}{4}\right) = \tfrac{1}{2}\ln(1 + \sin\tfrac{\pi}{2}) = \tfrac{1}{2}\ln 2.$$

$$F(0) = \tfrac{1}{2}\ln(1 + 0) = 0.$$

$$\int_0^{\pi/4} = \tfrac{1}{2}\ln 2 - 0 = \tfrac{1}{2}\ln 2. \quad \checkmark$$

$\frac{1}{2}\ln 2$ (shown)

Question 8 (OCR 4724, Jun 2015, Q6)

Worked Solution

(i) Use the quotient rule to show that the derivative of $\frac{\cos x}{\sin x}$ is $\frac{-1}{\sin^2 x}$.

Let $u = \cos x$, $v = \sin x$, so $u' = -\sin x$, $v' = \cos x$.

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$:

$$\frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}. \quad \checkmark$$

(using $\sin^2 x + \cos^2 x = 1$.)

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{\sqrt{1+\cos 2x}}{\sin x \sin 2x}\, dx = \frac{1}{2}(\sqrt{6} - \sqrt{2})$.

Step 1: Simplify $\sqrt{1+\cos 2x}$.

$1 + \cos 2x = 2\cos^2 x$, so $\sqrt{1+\cos 2x} = \sqrt{2}\cos x$.

Step 2: Simplify $\sin 2x$.

$\sin 2x = 2\sin x \cos x$.

Step 3: Rewrite the integrand.

$$\frac{\sqrt{2}\cos x}{\sin x \cdot 2\sin x \cos x} = \frac{\sqrt{2}}{2\sin^2 x} = \frac{1}{\sqrt{2}\sin^2 x} = \frac{1}{\sqrt{2}}\csc^2 x.$$

Step 4: Integrate.

Since $\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\frac{1}{\sin^2 x}$ (from part (i)), we have $\int \frac{1}{\sin^2 x}\, dx = -\frac{\cos x}{\sin x}$.

$$\int_{\pi/6}^{\pi/4} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sin^2 x}\, dx = \frac{1}{\sqrt{2}}\left[-\frac{\cos x}{\sin x}\right]_{\pi/6}^{\pi/4}.$$

Step 5: Apply limits.

$$-\frac{\cos(\pi/4)}{\sin(\pi/4)} = -\frac{1/\sqrt{2}}{1/\sqrt{2}} = -1.$$

$$-\frac{\cos(\pi/6)}{\sin(\pi/6)} = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}.$$

$$\frac{1}{\sqrt{2}}\left[(-1) - (-\sqrt{3})\right] = \frac{\sqrt{3}-1}{\sqrt{2}} = \frac{(\sqrt{3}-1)\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} = \frac{1}{2}(\sqrt{6}-\sqrt{2}). \quad \checkmark$$

$\frac{1}{2}(\sqrt{6} - \sqrt{2})$ (shown)

Question 9 (OCR 4724, Jan 2008, Q7)

Worked Solution

(i) Given that $A(\sin\theta + \cos\theta) + B(\cos\theta - \sin\theta) \equiv 4\sin\theta$, find A and B.

Expand the left side:

$$(A - B)\sin\theta + (A + B)\cos\theta \equiv 4\sin\theta + 0 \cdot \cos\theta.$$

Comparing coefficients:

$$A - B = 4 \quad \text{and} \quad A + B = 0.$$

Adding: $2A = 4 \Rightarrow A = 2$. Then $B = -A = -2$.

$A = 2, \quad B = -2$

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{4\sin\theta}{\sin\theta + \cos\theta}\, d\theta$, giving your answer in the form $a\pi - \ln b$.

Step 1: Rewrite the integrand using part (i).

Since $4\sin\theta = A(\sin\theta + \cos\theta) + B(\cos\theta - \sin\theta)$ with $A = 2$, $B = -2$:

$$\frac{4\sin\theta}{\sin\theta + \cos\theta} = A + \frac{B(\cos\theta - \sin\theta)}{\sin\theta + \cos\theta} = 2 + \frac{-2(\cos\theta - \sin\theta)}{\sin\theta + \cos\theta}.$$

Step 2: Integrate each part.

$\displaystyle\int A\, d\theta = A\theta = 2\theta.$

For the second part, note $\dfrac{d}{d\theta}(\sin\theta + \cos\theta) = \cos\theta - \sin\theta$, so:

$$\int \frac{B(\cos\theta - \sin\theta)}{\sin\theta + \cos\theta}\, d\theta = B\ln|\sin\theta + \cos\theta| = -2\ln|\sin\theta + \cos\theta|.$$

So:

$$F(\theta) = 2\theta - 2\ln|\sin\theta + \cos\theta|.$$

Step 3: Apply limits.

$$F\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{2} - 2\ln\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) = \tfrac{\pi}{2} - 2\ln\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) = \tfrac{\pi}{2} - 2\ln\sqrt{2}.$$

$$F(0) = 0 - 2\ln(0 + 1) = 0.$$

$$\int_0^{\pi/4} = \tfrac{\pi}{2} - 2\ln\sqrt{2} = \tfrac{\pi}{2} - \ln 2.$$

$$\frac{\pi}{2} - \ln 2$$

End of Worked Solutions