Question 1

(Jun 2005, Q4)

Worked Solution

(i) Show that $x = \tan\theta$ transforms $\int \frac{1}{(1+x^2)^2}\,dx$ to $\int \cos^2\theta\,d\theta$.

$x = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta$, so $dx = \sec^2\theta\,d\theta$.
Also $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$.

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\sec^4\theta}\cdot\sec^2\theta\,d\theta = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta \quad \checkmark$$

(ii) Hence find the exact value of $\int_0^1 \frac{1}{(1+x^2)^2}\,dx$.

Change limits: $x = 0 \Rightarrow \theta = 0$; $x = 1 \Rightarrow \theta = \pi/4$.

$$\int_0^{\pi/4} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\,d\theta = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/4}$$
$$= \frac{\pi}{8} + \frac{1}{4}$$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \frac{\pi}{8} + \frac{1}{4}$$

Question 2

(Jan 2006, Q6)

Worked Solution

(i) Show that $x = \sin^2\theta$ transforms $\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$ to $\int 2\sin^2\theta\,\mathrm{d}\theta$.

$x = \sin^2\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sin\theta\cos\theta.$

$\frac{x}{1-x} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$, so $\sqrt{\frac{x}{1-x}} = \tan\theta$ (taking positive root).

$$\int \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \int \tan\theta \cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta = \int \frac{\sin\theta}{\cos\theta}\cdot 2\sin\theta\cos\theta\,\mathrm{d}\theta = 2\int \sin^2\theta\,\mathrm{d}\theta \quad \checkmark$$

(ii) Hence find $\int_0^1 \sqrt{\frac{x}{1-x}}\,\mathrm{d}x$.

Limits: $x = 0 \Rightarrow \theta = 0$; $x = 1 \Rightarrow \theta = \pi/2$.

$$2\int_0^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = 2\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$$

$$\int_0^1 \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \frac{\pi}{2}$$

Question 3

(Jan 2008, Q10i)

Worked Solution

Use $x = \sin\theta$ to find the exact value of $\displaystyle\int_0^{1/2} \frac{1}{(1-x^2)^{3/2}}\,\mathrm{d}x$.

$x = \sin\theta \Rightarrow \mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. Also $(1-x^2)^{3/2} = \cos^3\theta$.

Limits: $x = 0 \Rightarrow \theta = 0$; $x = \frac{1}{2} \Rightarrow \theta = \pi/6$.

$$\int_0^{\pi/6} \frac{\cos\theta}{\cos^3\theta}\,\mathrm{d}\theta = \int_0^{\pi/6} \sec^2\theta\,\mathrm{d}\theta = \Big[\tan\theta\Big]_0^{\pi/6} = \tan\frac{\pi}{6} - 0 = \frac{1}{\sqrt{3}}$$

$$\int_0^{1/2} \frac{1}{(1-x^2)^{3/2}}\,\mathrm{d}x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Question 4

(Jun 2008, Q8)

Worked Solution

(i) Find A and B such that $\frac{2t}{(t+1)^2} \equiv \frac{A}{t+1} + \frac{B}{(t+1)^2}$.

$2t = A(t+1) + B$. At $t = -1$: $-2 = B$, so $B = -2$. Comparing t-coefficients: $A = 2$.

$A = 2,\ B = -2$

(ii) Show that $t = \sqrt{2x-1}$ transforms $\int \frac{1}{x+\sqrt{2x-1}}\,dx$ to $\int \frac{2t}{(t+1)^2}\,dt$.

$t = \sqrt{2x-1} \Rightarrow t^2 = 2x - 1 \Rightarrow x = \frac{t^2+1}{2}$, $dx = t\,dt$.

$x + \sqrt{2x-1} = \frac{t^2+1}{2} + t = \frac{t^2+2t+1}{2} = \frac{(t+1)^2}{2}$.

$$\int \frac{1}{x+\sqrt{2x-1}}\,dx = \int \frac{t\,dt}{(t+1)^2/2} = \int \frac{2t}{(t+1)^2}\,dt \quad \checkmark$$

(iii) Hence find the exact value of $\int_1^5 \frac{1}{x+\sqrt{2x-1}}\,dx$.

From (i): $\frac{2t}{(t+1)^2} = \frac{2}{t+1} - \frac{2}{(t+1)^2}$.

$$\int \frac{2t}{(t+1)^2}\,dt = 2\ln(t+1) + \frac{2}{t+1} + c$$

Limits: $x = 1 \Rightarrow t = 1$; $x = 5 \Rightarrow t = 3$.

$$\left[2\ln(t+1) + \frac{2}{t+1}\right]_1^3 = \left(2\ln 4 + \frac{1}{2}\right) - (2\ln 2 + 1) = 2\ln 2 - \frac{1}{2}$$

$$\int_1^5 \frac{1}{x+\sqrt{2x-1}}\,dx = 2\ln 2 - \frac{1}{2}$$

Question 5

(Jan 2009, Q5)

Worked Solution

(i) Show $u = \sqrt{x}$ transforms $\int \frac{1}{x(1+\sqrt{x})}\,dx$ to $\int \frac{2}{u(1+u)}\,du$.

$u = \sqrt{x} \Rightarrow x = u^2$, $dx = 2u\,du$.

$$\int \frac{1}{u^2(1+u)} \cdot 2u\,du = \int \frac{2}{u(1+u)}\,du \quad \checkmark$$

(ii) Hence find the exact value of $\int_1^9 \frac{1}{x(1+\sqrt{x})}\,dx$.

Partial fractions: $\frac{2}{u(1+u)} = \frac{2}{u} - \frac{2}{1+u}$.

Limits: $x = 1 \Rightarrow u = 1$; $x = 9 \Rightarrow u = 3$.

$$\int_1^3 \left(\frac{2}{u} - \frac{2}{1+u}\right) du = \Big[2\ln u - 2\ln(1+u)\Big]_1^3$$

$$= (2\ln 3 - 2\ln 4) - (2\ln 1 - 2\ln 2) = 2\ln 3 - 2\ln 4 + 2\ln 2 = 2\ln 3 - 2\ln 2 = \ln\frac{9}{4}$$

$$\int_1^9 \frac{1}{x(1+\sqrt{x})}\,dx = \ln\frac{9}{4}$$

Question 6

(Jan 2010, Q4)

Worked Solution

Use $u = 2 + \ln t$ to find the exact value of $\displaystyle\int_1^e \frac{1}{t(2+\ln t)^2}\,\mathrm{d}t$.

$u = 2 + \ln t \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1}{t}$, so $\dfrac{\mathrm{d}t}{t} = \mathrm{d}u$.

Limits: $t = 1 \Rightarrow u = 2$; $t = e \Rightarrow u = 3$.

$$\int_2^3 \frac{1}{u^2}\,\mathrm{d}u = \left[-\frac{1}{u}\right]_2^3 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$

$$\int_1^e \frac{1}{t(2+\ln t)^2}\,\mathrm{d}t = \frac{1}{6}$$

Question 7

(Jan 2011, Q5)

Worked Solution

$I = \displaystyle\int_2^5 \frac{5-x}{2+\sqrt{x-1}}\,\mathrm{d}x.$

(i) Show $u = \sqrt{x-1}$ transforms I to $\displaystyle\int_1^2 (4u - 2u^2)\,\mathrm{d}u$ and find the exact value.

$u = \sqrt{x-1} \Rightarrow x = u^2 + 1$, $\mathrm{d}x = 2u\,\mathrm{d}u$. Also $5 - x = 4 - u^2$.

Limits: $x = 2 \Rightarrow u = 1$; $x = 5 \Rightarrow u = 2$.

$$I = \int_1^2 \frac{4-u^2}{2+u}\cdot 2u\,\mathrm{d}u = \int_1^2 \frac{(2-u)(2+u)}{(2+u)}\cdot 2u\,\mathrm{d}u = \int_1^2 2u(2-u)\,\mathrm{d}u = \int_1^2 (4u-2u^2)\,\mathrm{d}u \quad ✓$$

$$\int_1^2 (4u - 2u^2)\,\mathrm{d}u = \left[2u^2 - \frac{2u^3}{3}\right]_1^2 = \left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right) = 6 - \frac{14}{3} = \frac{4}{3}$$

$I = \dfrac{4}{3}$

(ii)(a) Simplify $(2+\sqrt{x-1})(2-\sqrt{x-1})$.

$$(2+\sqrt{x-1})(2-\sqrt{x-1}) = 4 - (x-1) = 5 - x$$

$5 - x$

(ii)(b) By multiplying numerator and denominator of $\dfrac{5-x}{2+\sqrt{x-1}}$ by $2 - \sqrt{x-1}$, find the exact value.

$$\frac{5-x}{2+\sqrt{x-1}} = \frac{(5-x)(2-\sqrt{x-1})}{5-x} = 2 - \sqrt{x-1}$$

$$I = \int_2^5 (2-\sqrt{x-1})\,\mathrm{d}x = \left[2x - \frac{2}{3}(x-1)^{3/2}\right]_2^5 = \left(10 - \frac{16}{3}\right) - \left(4 - \frac{2}{3}\right) = 6 - \frac{14}{3} = \frac{4}{3} \quad ✓$$

Question 8

(Jan 2013, Q6)

Worked Solution

Use $u = 2x + 1$ to evaluate $\displaystyle\int_0^{1/2} \frac{4x-1}{(2x+1)^5}\,\mathrm{d}x$.

$u = 2x + 1 \Rightarrow x = \dfrac{u-1}{2}$, $\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$.

$4x - 1 = 2(u-1) - 1 = 2u - 3$.

Limits: $x = 0 \Rightarrow u = 1$; $x = \frac{1}{2} \Rightarrow u = 2$.

$$\int_1^2 \frac{2u-3}{u^5} \cdot \frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\int_1^2 (2u^{-4} - 3u^{-5})\,\mathrm{d}u = \frac{1}{2}\left[-\frac{2}{3u^3} + \frac{3}{4u^4}\right]_1^2$$

At $u = 2$: $-\dfrac{2}{24} + \dfrac{3}{64} = -\dfrac{1}{12} + \dfrac{3}{64} = \dfrac{-16 + \frac{9}{4}}{\ldots}$

Let me compute carefully: $-\dfrac{2}{3(8)} + \dfrac{3}{4(16)} = -\dfrac{1}{12} + \dfrac{3}{64} = \dfrac{-16}{192} + \dfrac{9}{192} = -\dfrac{7}{192}$

At $u = 1$: $-\dfrac{2}{3} + \dfrac{3}{4} = \dfrac{-8+9}{12} = \dfrac{1}{12}$

$$= \frac{1}{2}\left(-\frac{7}{192} - \frac{1}{12}\right) = \frac{1}{2}\left(-\frac{7}{192} - \frac{16}{192}\right) = \frac{1}{2}\cdot\left(-\frac{23}{192}\right) = -\frac{23}{384}$$

$$\int_0^{1/2} \frac{4x-1}{(2x+1)^5}\,\mathrm{d}x = -\frac{23}{384}$$

Question 9

(Jun 2016, Q6)

Worked Solution

Use $u = x^2 - 2$ to find $\displaystyle\int \frac{6x^3 + 4x}{\sqrt{x^2 - 2}}\,\mathrm{d}x$.

$u = x^2 - 2 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$, so $x\,\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$. Also $x^2 = u + 2$.

$$\frac{6x^3 + 4x}{\sqrt{x^2 - 2}} = \frac{x(6x^2 + 4)}{\sqrt{u}} = \frac{(6(u+2)+4)}{\sqrt{u}} \cdot x = \frac{6u + 16}{\sqrt{u}} \cdot x$$

$$\int \frac{6x^3 + 4x}{\sqrt{x^2 - 2}}\,\mathrm{d}x = \int \frac{6u + 16}{\sqrt{u}} \cdot \frac{\mathrm{d}u}{2} = \frac{1}{2}\int \left(6u^{1/2} + 16u^{-1/2}\right)\mathrm{d}u$$

$$= \frac{1}{2}\left(4u^{3/2} + 32u^{1/2}\right) + c = 2u^{3/2} + 16u^{1/2} + c$$

Back-substitute $u = x^2 - 2$:

$$= 2(x^2 - 2)^{3/2} + 16(x^2 - 2)^{1/2} + c$$

$$\int \frac{6x^3 + 4x}{\sqrt{x^2 - 2}}\,\mathrm{d}x = 2(x^2 - 2)^{3/2} + 16\sqrt{x^2 - 2} + c$$

Question 10

(Jun 2017, Q9)

Worked Solution

Use $u = 1 + \ln x + x$ to find $\displaystyle\int \frac{3(x+1)(1-\ln x - x)}{x(1+\ln x + x)}\,\mathrm{d}x$.

$u = 1 + \ln x + x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x} + 1 = \dfrac{x+1}{x}$, so $\dfrac{x+1}{x}\,\mathrm{d}x = \mathrm{d}u$.

Note the numerator: $3(x+1)(1-\ln x - x) = -3(x+1)(\ln x + x - 1)$.

Also $\ln x + x - 1 = u - 2$, and the fraction:

$$\frac{3(x+1)(1-\ln x - x)}{x(1+\ln x + x)} = \frac{-3(x+1)(u-2)}{xu}$$

$$\int \frac{3(x+1)(1-\ln x - x)}{x(1+\ln x + x)}\,\mathrm{d}x = \int \frac{-3(u-2)}{u}\,\mathrm{d}u = -3\int \left(1 - \frac{2}{u}\right)\mathrm{d}u$$
$$= -3u + 6\ln|u| + c = -3(1+\ln x + x) + 6\ln|1+\ln x + x| + c$$

Since the constant absorbs -3:

$$\int \frac{3(x+1)(1-\ln x - x)}{x(1+\ln x + x)}\,\mathrm{d}x = 6\ln(1+\ln x + x) - 3(1+\ln x + x) + c$$

End of Worked Solutions