# Integration By Substitution Exam Questions MS (From OCR 4724)

## Q1, (Jun 2005, Q4)

| | | |
|---|---|---|
| (i) $dx = \sec^2\theta \, d\theta$ AEF | M1 | Attempt to connect $dx, d\theta$ (not $dx = d\theta$) |
| | A1 | For $dx = \sec^2\theta \, d\theta$ or equiv correctly used |
| Indefinite integral = $\int \cos^2\theta \, d\theta$ | A1 **3** | With at least one intermed step **AG** |
| (ii) $= k\int +/- 1 +/- \cos 2\theta \, d\theta$ | M1 | "Satis" attempt to change to double angle |
| $\frac{1}{2}[\theta + \frac{1}{2}\sin 2\theta]$ | A1 | Correct attempt + correct integration |
| Limits = $\frac{1}{4}\pi$(accept 45) and 0 | M1 | New limits for $\theta$ or resubstituting |
| $(\pi + 2)/8$ AEF | A1 **4** | Ignore decimals after correct answer |
| | | **7** |
| | | Single 'parts' + $\sin^2\theta = 1 - \cos^2\theta$ acceptable |

## Q2, (Jan 2006, Q6)

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|---|---|---|---|
| **(i)** Attempt to connect $dx, d\theta$ | M1 | | But not $dx = d\theta$ |
| $dx = 2\sin\theta\cos\theta \, d\theta$ | A1 | | AEF |
| $\sqrt{\frac{x}{1-x}} = \frac{\sin\theta}{\cos\theta}$ | B1 | | Ignore any references to $\pm$. |
| Reduction to $\int 2\sin^2\theta \, d\theta$ | A1 | **4** | **AG** WWW |
| **(ii)** $\sin^2\theta = k(+/-1+/-\cos 2\theta)$ | M1 | | Attempt to change $(2)\sin^2\theta$ into $f(\cos 2\theta)$ |
| $2\sin^2\theta = 1 - \cos 2\theta$ | A1 | | Correct attempt |
| $\int \cos 2\theta \, d\theta = \frac{1}{2}\sin 2\theta$ | B1 | | Seen anywhere in this part |
| Attempting to change limits | M1 | | <u>Or</u> Attempting to resubstitute; Accept degrees |
| $\frac{1}{2}\pi$ | A1 | **5** | |
| <u>Alternatively</u> Parts once & use | | | |
| $\cos^2\theta = 1 - \sin^2\theta$ | (M2) | | Instead of the M1 A1 B1 |
| $\frac{1}{2}(\theta - \sin\theta\cos\theta)$ | (A1) | | Then the final M1 A1 for use of limits |

## Q3, (Jan 2008, Q10i)

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|---|---|---|---|
| $(1-x^2)^{\frac{3}{2}} \to \cos^3\theta$ | B1 | | May be implied by $\int \sec^2\theta \, d\theta$ |
| $dx \to \cos\theta \, d\theta$ | B1 | | |
| $\frac{1}{(1-x^2)^{\frac{3}{2}}} dx \to \sec^2\theta \, (d\theta)$ or $\frac{1}{\cos^2\theta}(d\theta)$ | B1 | | |
| $\int \sec^2\theta \, (d\theta) = \tan\theta$ | B1 | | |
| Attempt change of limits (expect 0 & $\frac{1}{6}\pi / 30$) | M1 | | Use with $f(\theta)$; or re-subst & use 0 & $\frac{1}{2}$ |
| $\frac{1}{\sqrt{3}}$ AEF | A1 | **6** | Obtained with no mention of 30 anywhere |

**Q4, (Jun 2008, Q8)**

| | | | |
|---|---|---|---|
| **(i)** | $A(t+1)+B=2t$ | **M1** | Beware: correct values for $A$ and/or $B$ can be ... |
| | $A=2$ | **A1** | … obtained from a wrong identity |
| | $B=-2$ | **A1** | Alt method: subst suitable values into given… |
| | | | …expressions |
| | | **3** | |
| **(ii)** | Attempt to connect d$x$ and d$t$ | **M1** | But not just d$x$ = d$t$. As **AG**, look carefully. |
| | d$x$ = $t$ d$t$ s.o.i. AEF | **A1** | |
| | $x+\sqrt{2x-1}\to\dfrac{t^2+1}{2}+t=\dfrac{(t+1)^2}{2}$ s.o.i. | **B1** | Any wrong working invalidates |
| | $\displaystyle\int\frac{2t}{(t+1)^2}\,\mathrm{d}t$ | **A1** | **AG** WWW The 'd$t$' must be present |
| | | **4** | |
| **(iii)** | $\displaystyle\int\frac{1}{t+1}\,\mathrm{d}t=\ln(t+1)$ | **B1** | Or parts $u=2t, \mathrm{d}v=(t+1)^{-2}$ or subst $u=t+1$ |
| | $\displaystyle\int\frac{1}{(t+1)^2}\,\mathrm{d}t=-\frac{1}{t+1}$ | **B1** | |
| | Attempt to change limits (expect 1 & 3) and use f($t$) | **M1** | or re-substitute and use 1 and 5 on g($x$) |
| | $\ln 4-\dfrac{1}{2}$ | **A1** | AEF (like terms amalgamated); if A0 A0 in (i), then final A0 |
| | | **4** | |

**Q5, (Jan 2009, Q5)**

| | | | |
|---|---|---|---|
| **(i)** | Attempt to connect d$u$ and d$x$, find $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ or $\dfrac{\mathrm{d}x}{\mathrm{d}u}$ | M1 | But not e.g. d$u$ = d$x$ |
| | Any correct relationship, however used, such as d$x$ = 2$u$ d$u$ | A1 | or $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2}x^{-½}$ |
| | Subst with clear reduction ($\geq 1$ inter step) to **AG** | A1 **(3)** | WWW |
| **(ii)** | Attempt partial fractions | M1 | |
| | $\dfrac{2}{u}-\dfrac{2}{1+u}$ | A1 | |
| | √ $A\ln u+B\ln(1+u)$ | √A1 | Based on $\dfrac{A}{u}+\dfrac{B}{1+u}$ |
| | Attempt integ, change limits & use on f($u$) | M1 | or re-subst & use 1 & 9 |
| | $\ln\dfrac{9}{4}$ AEexactF (e.g. 2 ln 3 –2 ln 4 + 2 ln 2) | A1 **(5)** | Not involving ln 1 |
| | | **8** | |

**Q6, (Jan 2010, Q4)**

| | | |
|---|---|---|
| Attempt to connect $\mathrm{d}u$ and $\mathrm{d}t$ or find $\frac{\mathrm{d}u}{\mathrm{d}t}$ or $\frac{\mathrm{d}t}{\mathrm{d}u}$ | M1 | not $\mathrm{d}u = \mathrm{d}t$ but no accuracy |
| $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$ or $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{t}$ or $\mathrm{d}t = \mathrm{e}^{u-2}\,\mathrm{d}u$ or $\frac{\mathrm{d}t}{\mathrm{d}u} = \mathrm{e}^{u-2}$ | A1 | |
| Indef int $\rightarrow \int \frac{1}{u^2}(\mathrm{d}u)$ | A1 | no $t$ or $\mathrm{d}t$ in evidence |
| $= -\frac{1}{\mathrm{u}}$ | A1 | |
| Attempt to change limits if working with $\mathrm{f}(u)$ | M1 | or re-subst & use 1 and e |
| $\frac{1}{6}$ ISW | A1 | ln e must be changed to 1, ln 1 to 0 |
| | **6** | |

**Q7, (Jan 2011, Q5)**

| | | | |
|---|---|---|---|
| **(i)** Attempt to connect $\mathrm{d}x$ and $\mathrm{d}u$ | M1 | | Including $\frac{\mathrm{d}u}{\mathrm{d}x} =$ or $\mathrm{d}u = \ldots\mathrm{d}x$ ; not $\mathrm{d}x = \mathrm{d}u$ |
| $5 - x = 4 - u^2$ | B1 | | perhaps in conjunction with next line |
| Show $\int \frac{4-u^2}{2+u}.2u\,\mathrm{d}u$ reduced to $\int 4u - 2u^2\,\mathrm{d}u$ **AG** | A1 | | In a fully satisfactory & acceptable manner |
| Clear explanation of why limits change | B1 | | e.g. when $x = 2$, $u = 1$ <u>and</u> when $x = 5$, $u = 2$ |
| $\frac{4}{3}$ | B1 | **5** | not dependent on any of first 4 marks |
| **(ii)(a)** $5 - x$ | *B1 | **1** | Accept $4 - x - 1 = 5 - x$ (this is not **AG**) |
| **(b)** Show reduction to $2 - \sqrt{x-1}$ | dep*B1 | | |
| $\int \sqrt{x-1}\,\mathrm{d}x = \frac{2}{3}(x-1)^{\frac{3}{2}}$ | B1 | | Indep of other marks, seen anywhere in (b) |
| $\left(10 - \frac{2}{3}.8\right) - \left(4 - \frac{2}{3}\right) = \frac{4}{3}$ or $4\frac{2}{3} - 3\frac{1}{3} = \frac{4}{3}$ | B1 | **3** | Working must be shown |
| | | **9** | |

**Q8, (Jan 2013, Q6)**

| | | | |
|---|---|---|---|
| Attempt diff to connect d$u$ & d$x$ | M1 | or find $\frac{du}{dx}$ or $\frac{dx}{du}$ | |
| Correct result e.g. $\frac{du}{dx} = 2$ or $du = 2\,dx$ | A1 | | |
| Indef integ in terms of $u = \frac{1}{2}\int \frac{2u-3}{u^5}(du)$ | A1 | Must be completely in terms of $u$. | |
| Integrate to $\frac{u^{-3}}{-3} - \frac{3u^{-4}}{-8}$ <u>oe</u> | A1A1 | or (using 'by parts') $\frac{(2u-3)u^{-4}}{-8} - \frac{u^{-3}}{12}$ | Award B1,B1 for $\frac{4u^{-3}}{-3} - \frac{3u^{-4}}{-2}$ or for $\frac{2u^{-3}}{-3} - \frac{3u^{-4}}{-4}$ or for $\frac{(2u-3)u^{-4}}{-2} - \frac{u^{-3}}{3}$ or for $\frac{(2u-3)u^{-4}}{-4} - \frac{u^{-3}}{6}$ |
| Use correct variable & correct values for limits | M1 | Provided minimal attempt at $\int f(u)du$ made | |
| $= \frac{-23}{384}$ oe $(-0.059895 \ldots..)$ | A1 | Accept decimal answer only if minimum of first 3 marks scored | |
| [ISW,e.g. changing to $\frac{23}{384}$] | | | |
| | **[7]** | | |

**Q9, (Jun 2016, Q6)**

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|---|---|---|---|
| $\frac{du}{dx} = 2x$ oe or $\frac{dx}{du} = \frac{1}{2}(u \pm 2)^{-1/2}$ oe | **M1** | | |
| $\frac{Ax^2 + B}{2}$ or better from replacing d$x$ NB $\frac{6x^3 + 4x}{2x} = \frac{6x^2 + 4}{2}$ | **M1** | | or substitution of $x = (u \pm 2)^{1/2}$ in denominator from $\frac{dx}{du}$ |
| substitution of $x^2 = u \pm 2$ or $x = (u \pm 2)^{1/2}$ in numerator | **M1** | **NB** $3(u+2)+2$ or $3(u+2)^{3/2} + 2(u+2)^{1/2}$ | |
| $\int (\frac{3u+8}{\sqrt{u}})[du]$ oe | **A1** | $\frac{3(u+2)+2}{\sqrt{u}}$ or better | |
| $\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{8u^{\frac{1}{2}}}{\frac{1}{2}}$ oe | **A1** | or $6u^{3/2} + 16u^{1/2} - 4u^{3/2}$ from integration by parts | |
| $2(x^2 - 2)^{\frac{3}{2}} + 16(x^2 - 2)^{\frac{1}{2}} + c$ cao | **A1** **[6]** | allow $2(x^2-2)^{1/2}(x^2+6) + c$ for final mark, **A0** if d$u$ not seen at some stage in the integral | must see constant of integration here or in previous line and coefficients must be simplified for final **A1** |

**Q10, (Jun 2017, Q9)**

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|---|---|---|---|
| $\frac{du}{dx} = 1 + \frac{1}{x}$ | **B1** | | |
| $x + \ln x = \pm u \pm 1$ oe substituted into the numerator | **M1*** | allow slip in substitution | |
| dx replaced by *their* $\left(\frac{1}{1/x + 1}\right)$[d$u$] in integrand oe | **M1*** | | |
| $\int \left(\frac{3(1-(u-1))}{u}\right)[du]$ oe | **A1** | may be simplified | $\int \left(\frac{6}{u} - 3\right) du$ |
| $A \ln u + Bu$ $(+ c)$ | **M1dep*** | following $\int \left(\frac{A}{u} + B\right) du$ | |
| $6\ln(1 + \ln x + x) - 3(1 + \ln x + x) + c$ oe isw | **A1** | | if d$u$ and/or $\int$ and/or $+ c$ not seen at some stage, withhold the final **A1** |
| | **[6]** | | |