Question 1

Worked Solution

Using $u = 2+\sqrt{2x+1}$, find the exact value of $\int_0^4 \frac{1}{2+\sqrt{2x+1}}\,\mathrm{d}x$, giving your answer in the form $A + 2\ln B$.

$u = 2+\sqrt{2x+1} \Rightarrow \sqrt{2x+1} = u-2 \Rightarrow 2x+1 = (u-2)^2$.

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{2x+1}} = \frac{1}{u-2}$, so $\mathrm{d}x = (u-2)\,\mathrm{d}u$.

Limits: $x = 0 \Rightarrow u = 3$; $x = 4 \Rightarrow u = 5$.

$$\int_3^5 \frac{(u-2)}{u}\,\mathrm{d}u = \int_3^5 \left(1 - \frac{2}{u}\right)\mathrm{d}u = \Big[u - 2\ln u\Big]_3^5$$

$$= (5 - 2\ln 5) - (3 - 2\ln 3) = 2 - 2\ln 5 + 2\ln 3 = 2 + 2\ln\frac{3}{5}$$

$$\int_0^4 \frac{1}{2+\sqrt{2x+1}}\,\mathrm{d}x = 2 + 2\ln\frac{3}{5}, \quad \text{so } A = 2,\ B = \frac{3}{5}.$$

Question 2

Worked Solution

Using $u = \cos x + 1$**, show that** $\displaystyle\int_0^{\pi/2} e^{\cos x+1} \sin x \, dx = e(e-1)$**.**

$u = \cos x + 1 \Rightarrow \dfrac{du}{dx} = -\sin x$, so $\sin x \, dx = -du$.

Limits: $x = 0 \Rightarrow u = 2$; $x = \pi/2 \Rightarrow u = 1$.

$$\int_2^1 e^u(-du) = \int_1^2 e^u \, du = \Big[e^u\Big]_1^2 = e^2 - e = e(e-1) \quad \checkmark$$

Question 3

Worked Solution

(i) Given $y > 0$, find $\int \frac{3y-4}{y(3y+2)}\,dy$.

Partial fractions: $\frac{3y-4}{y(3y+2)} = \frac{A}{y} + \frac{B}{3y+2}$.

$3y - 4 = A(3y+2) + By$. At $y = 0$: $-4 = 2A \Rightarrow A = -2$. At $y = -\frac{2}{3}$: $-6 = -\frac{2B}{3} \Rightarrow B = 9$.

$$\int \left(-\frac{2}{y} + \frac{9}{3y+2}\right) dy = -2\ln y + 3\ln(3y+2) + c$$

$$\int \frac{3y-4}{y(3y+2)}\,dy = -2\ln y + 3\ln(3y+2) + c$$

(ii)(a) Use $x = 4\sin^2\theta$ to show that $\int_0^3 \sqrt{\frac{x}{4-x}}\,dx = \lambda \int_0^{\pi/3} \sin^2\theta\,d\theta$, finding λ.

$x = 4\sin^2\theta \Rightarrow dx = 8\sin\theta\cos\theta\,d\theta$.

$4 - x = 4\cos^2\theta$, so $\sqrt{\frac{x}{4-x}} = \sqrt{\frac{4\sin^2\theta}{4\cos^2\theta}} = \tan\theta$.

Limits: $x = 0 \Rightarrow \theta = 0$; $x = 3 \Rightarrow \sin^2\theta = \frac{3}{4} \Rightarrow \theta = \pi/3$.

$$\int_0^{\pi/3} \tan\theta \cdot 8\sin\theta\cos\theta\,d\theta = \int_0^{\pi/3} \frac{\sin\theta}{\cos\theta} \cdot 8\sin\theta\cos\theta\,d\theta = 8\int_0^{\pi/3} \sin^2\theta\,d\theta$$

$\lambda = 8$

(ii)(b) Hence find $\int_0^3 \sqrt{\frac{x}{4-x}}\,dx$, giving your answer in the form $a\pi + b$.

$$8\int_0^{\pi/3} \sin^2\theta\,d\theta = 8\int_0^{\pi/3} \frac{1-\cos 2\theta}{2}\,d\theta = 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= 4\left(\frac{\pi}{3} - \frac{\sin(2\pi/3)}{2}\right) = 4\left(\frac{\pi}{3} - \frac{\sqrt{3}/2}{2}\right) = \frac{4\pi}{3} - \sqrt{3}$$

$$\int_0^3 \sqrt{\frac{x}{4-x}}\,dx = \frac{4\pi}{3} - \sqrt{3}$$

Question 4

Worked Solution

(a) Use $x = u^2$, $u > 0$, to show that $\int \frac{1}{x(2\sqrt{x}-1)}\,\mathrm{d}x = \int \frac{2}{u(2u-1)}\,\mathrm{d}u$.

$x = u^2 \Rightarrow \mathrm{d}x = 2u\,\mathrm{d}u$, $\sqrt{x} = u$.

$$\int \frac{2u\,\mathrm{d}u}{u^2(2u-1)} = \int \frac{2}{u(2u-1)}\,\mathrm{d}u \quad \checkmark$$

(b) Hence show that $\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,\mathrm{d}x = 2\ln\left(\frac{a}{b}\right)$ where a, b are integers.

Partial fractions: $\frac{2}{u(2u-1)} = \frac{A}{u} + \frac{B}{2u-1}$.

$2 = A(2u-1) + Bu$. At $u = 0$: $A = -2$. At $u = \frac{1}{2}$: $B = 4$.

Limits: $x = 1 \Rightarrow u = 1$; $x = 9 \Rightarrow u = 3$.

$$\int_1^3 \left(-\frac{2}{u} + \frac{4}{2u-1}\right)\mathrm{d}u = \Big[-2\ln u + 2\ln(2u-1)\Big]_1^3$$

$$= (-2\ln 3 + 2\ln 5) - (0+0) = 2\ln 5 - 2\ln 3 = 2\ln\frac{5}{3}$$

$$\int_1^9 \frac{1}{x(2\sqrt{x}-1)}\,\mathrm{d}x = 2\ln\left(\frac{5}{3}\right), \quad a = 5,\ b = 3.$$

Question 5

Worked Solution

Show that $\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x = \frac{32}{15}(2+\sqrt{2})$**.**

Let $u = \sqrt{x+2}$, so $u^2 = x+2$, $x = u^2 - 2$, $\mathrm{d}x = 2u\,\mathrm{d}u$.

Limits: $x = 0 \Rightarrow u = \sqrt{2}$; $x = 2 \Rightarrow u = 2$.

$$\int_{\sqrt{2}}^{2} 2(u^2-2)\cdot u \cdot 2u\,\mathrm{d}u = 4\int_{\sqrt{2}}^{2}(u^4 - 2u^2)\,\mathrm{d}u$$
$$= 4\left[\frac{u^5}{5} - \frac{2u^3}{3}\right]_{\sqrt{2}}^{2}$$

At $u = 2$: $\frac{32}{5} - \frac{16}{3} = \frac{96-80}{15} = \frac{16}{15}$.

At $u = \sqrt{2}$: $\frac{(\sqrt{2})^5}{5} - \frac{2(\sqrt{2})^3}{3} = \frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} = 4\sqrt{2}\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{8\sqrt{2}}{15}$.

$$= 4\left(\frac{16}{15} - \left(-\frac{8\sqrt{2}}{15}\right)\right) = 4\cdot\frac{16+8\sqrt{2}}{15} = \frac{4\cdot 8(2+\sqrt{2})}{15} = \frac{32(2+\sqrt{2})}{15} \quad \checkmark$$

Question 6

Worked Solution

Curve: $y = \sqrt{(3-x)(x+1)}$, $0 \le x \le 3$. Region R bounded by curve, axes.

(a) Use $x = 1 + 2\sin\theta$ to show that $\int_0^3 \sqrt{(3-x)(x+1)}\,\mathrm{d}x = k\int_{-\pi/6}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta$, finding k.

$x = 1 + 2\sin\theta \Rightarrow \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$.

$3 - x = 2 - 2\sin\theta = 2(1-\sin\theta)$, $\quad x + 1 = 2 + 2\sin\theta = 2(1+\sin\theta)$.

$(3-x)(x+1) = 4(1-\sin^2\theta) = 4\cos^2\theta$.

Limits: $x = 0 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = -\pi/6$; $\quad x = 3 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$.

$$\int_{-\pi/6}^{\pi/2} 2\cos\theta \cdot 2\cos\theta\,\mathrm{d}\theta = 4\int_{-\pi/6}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta$$

$k = 4$

(b) Hence find the exact area of R.

$$4\int_{-\pi/6}^{\pi/2} \cos^2\theta\,\mathrm{d}\theta = 4\int_{-\pi/6}^{\pi/2} \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/6}^{\pi/2}$$

At $\theta = \pi/2$: $\frac{\pi}{2} + 0 = \frac{\pi}{2}$.

At $\theta = -\pi/6$: $-\frac{\pi}{6} + \frac{\sin(-\pi/3)}{2} = -\frac{\pi}{6} - \frac{\sqrt{3}}{4}$.

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right) = 2\left(\frac{\pi}{2} + \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) = 2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

$$\text{Area} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

End of Worked Solutions