Integration By Substitution (Sheet 2)

Q1.

Question Number	Scheme		Marks
	$\int_0^4 \frac{1}{2+\sqrt{(2x+1)}}\,dx$, $u = 2+\sqrt{(2x+1)}$		
	$\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = u-2$	Either $\frac{du}{dx} = \pm K(2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = \pm\lambda(u-2)$	M1
		Either $\frac{du}{dx} = (2x+1)^{-\frac{1}{2}}$ or $\frac{dx}{du} = (u-2)$	A1
	$\left\{\int \frac{1}{2+\sqrt{(2x+1)}}\,dx\right\} = \int \frac{1}{u}(u-2)\,du$	Correct substitution (Ignore integral sign and du).	A1
	$= \int \left(1-\frac{2}{u}\right)du$	An attempt to divide each term by u.	dM1
		$\pm Au \pm B\ln u$	ddM1
	$= u - 2\ln u$	$u - 2\ln u$	A1 ft
	$\left\{\text{So } [u-2\ln u]_3^5\right\} = (5-2\ln 5)-(3-2\ln 3)$	Applies limits of 5 and 3 in u or 4 and 0 in x in their integrated function and subtracts the correct way round.	M1
	$= 2+2\ln\left(\frac{3}{5}\right)$	$2+2\ln\left(\frac{3}{5}\right)$	A1 cao cso
			[8] 8

Notes for Question

M1: Also allow $du = \pm\lambda\frac{1}{(u-2)}dx$ or $(u-2)du = \pm\lambda\,dx$

Note: The expressions must contain du and dx. They can be simplified or un-simplified.

A1: Also allow $du = \frac{1}{(u-2)}dx$ or $(u-2)du = \pm\lambda\,dx$

Note: The expressions must contain du and dx. They can be simplified or un-simplified.

A1: $\int \frac{1}{u}(u-2)\,du$. (Ignore integral sign and du).

dM1: An attempt to divide each term by u.
Note that this mark is dependent on the previous M1 mark being awarded.
Note that this mark can be implied by later working.

ddM1: $\pm Au \pm B\ln u$, $A \neq 0$, $B \neq 0$
Note that this mark is dependent on the two previous M1 marks being awarded.

A1ft: $u - 2\ln u$ or $\pm Au \pm B\ln u$ being correctly followed through, $A \neq 0$, $B \neq 0$

M1: Applies limits of 5 and 3 in u or 4 and 0 in x in their integrated function and subtracts the correct way round.

A1: cso and cao. $2+2\ln\left(\frac{3}{5}\right)$ or $2+2\ln(0.6)$, $\left(= A+2\ln B, \text{ so } A=2, B=\frac{3}{5}\right)$

Note: $2-2\ln\left(\frac{3}{5}\right)$ is A0.

Q2.

Question Number	Scheme		Marks	
	$\frac{du}{dx} = -\sin x$		B1	
	$\int \sin x\, e^{\cos x+1} dx = -\int e^u du$		M1 A1	
	$= -e^u$	ft sign error	A1ft	
	$= -e^{\cos x+1}$			
	$\left[-e^{\cos x+1}\right]_0^{\frac{\pi}{2}} = -e^1 - \left(-e^2\right)$	or equivalent with u	M1	
	$= e(e-1)$ *	cso	A1	(6)
				[6]

Q3.

Question Number	Scheme	Notes	Marks
	(i) $\int \frac{3y-4}{y(3y+2)} dy,\ y>0$, (ii) $\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)} dx,\ x = 4\sin^2\theta$		
(i) Way 1	$\frac{3y-4}{y(3y+2)} \equiv \frac{A}{y} + \frac{B}{(3y+2)} \Rightarrow 3y-4 = A(3y+2) + By$	See notes	M1
	$y=0 \Rightarrow -4 = 2A \Rightarrow A = -2$	At least one of their $A=-2$ or their $B=9$	A1
	$y = -\frac{2}{3} \Rightarrow -6 = -\frac{2}{3}B \Rightarrow B = 9$	Both their $A=-2$ and their $B=9$	A1
	$\int \frac{3y-4}{y(3y+2)} dy = \int \frac{-2}{y} + \frac{9}{(3y+2)} dy$	Integrates to give at least one of either $\frac{A}{y} \to \pm\lambda \ln y$ or $\frac{B}{(3y+2)} \to \pm\mu \ln(3y+2)$; $A \neq 0,\ B \neq 0$	M1
		At least one term correctly followed through from their A or from their B	A1 ft
	$= -2\ln y + 3\ln(3y+2)\ \{+c\}$	$-2\ln y + 3\ln(3y+2)$ or $-2\ln y + 3\ln(y+\frac{2}{3})$ with correct bracketing, simplified or un-simplified. Can apply isw.	A1 cao
			[6]
(ii) (a) Way 1	$\{x = 4\sin^2\theta \Rightarrow\}\ \frac{dx}{d\theta} = 8\sin\theta\cos\theta$ or $\frac{dx}{d\theta} = 4\sin 2\theta$ or $dx = 8\sin\theta\cos\theta\, d\theta$		B1
	$\int \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}.\, 8\sin\theta\cos\theta\ \{d\theta\}$ or $\int \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}.\, 4\sin 2\theta\ \{d\theta\}$		M1
	$= \int \underline{\tan\theta}.\, 8\sin\theta\cos\theta\ \{d\theta\}$ or $\int \underline{\tan\theta}.\, 4\sin 2\theta\ \{d\theta\}$	$\sqrt{\left(\frac{x}{4-x}\right)} \to \pm K\tan\theta$ or $\pm K\left(\frac{\sin\theta}{\cos\theta}\right)$	<u>M1</u>
	$= \int 8\sin^2\theta\, d\theta$	$\int 8\sin^2\theta\, d\theta$ including $d\theta$	A1
	$3 = 4\sin^2\theta$ or $\frac{3}{4} = \sin^2\theta$ or $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$; $\{x = 0 \to \theta = 0\}$	Writes down a correct equation involving $x = 3$ leading to $\theta = \frac{\pi}{3}$ and no incorrect work seen regarding limits	B1
			[5]
(ii) (b)	$= \{8\}\int\left(\frac{1-\cos 2\theta}{2}\right) d\theta\ \left\{= \int (4 - 4\cos 2\theta) d\theta\right\}$	Applies $\cos 2\theta = 1 - 2\sin^2\theta$ to their integral. (See notes)	M1
	$= \{8\}\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\ \{= 4\theta - 2\sin 2\theta\}$	For $\pm\alpha\theta \pm \beta\sin 2\theta,\ \alpha, \beta \neq 0$	M1
		$\sin^2\theta \to \left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)$	A1
	$\left\{\int_0^{\frac{\pi}{3}} 8\sin^2\theta\, d\theta = 8\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}} = 8\left(\left(\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right) - (0+0)\right)\right\}$		
	$= \frac{4}{3}\pi - \sqrt{3}$	"two term" exact answer of e.g. $\frac{4}{3}\pi - \sqrt{3}$ or $\frac{1}{3}\left(4\pi - 3\sqrt{3}\right)$	A1 o.e.
			[4]
			15

	Question	Notes
(i)	1st M1	Writing $\frac{3y-4}{y(3y+2)} \equiv \frac{A}{y} + \frac{B}{(3y+2)}$ and a complete method for finding the value of at least one of their A or their B.
	Note	M1A1 can be implied *for writing down* either $\frac{3y-4}{y(3y+2)} \equiv \frac{-2}{y} + \frac{\text{their } B}{(3y+2)}$ or $\frac{3y-4}{y(3y+2)} \equiv \frac{\text{their } A}{y} + \frac{9}{(3y+2)}$ with no working.
	Note	Correct bracketing is not necessary for the penultimate A1ft, but is required for the final A1 in (i)
	Note	Give 2nd M0 for $\frac{3y-4}{y(3y+2)}$ going directly to $\pm\alpha\ln(3y^2+2y)$
	Note	...but allow 2nd M1 for either $\frac{M(6y+2)}{3y^2+2y} \to \pm\alpha\ln(3y^2+2y)$ or $\frac{M(3y+1)}{3y^2+2y} \to \pm\alpha\ln(3y^2+2y)$
(ii)(a)	1st M1	Substitutes $x = 4\sin^2\theta$ and their dx (from their correctly rearranged $\frac{dx}{d\theta}$) into $\int\sqrt{\left(\frac{x}{4-x}\right)}\,dx$
	Note	$dx \neq \lambda d\theta$. For example $dx \neq d\theta$
	Note	Allow substituting $dx = 4\sin 2\theta$ for the 1st M1 after a correct $\frac{dx}{d\theta} = 4\sin 2\theta$ or $dx = 4\sin 2\theta\, d\theta$
	2nd M1	Applying $x = 4\sin^2\theta$ to $\sqrt{\left(\frac{x}{4-x}\right)}$ to give $\pm K\tan\theta$ or $\pm K\left(\frac{\sin\theta}{\cos\theta}\right)$
	Note	Integral sign is not needed for this mark.
	1st A1	Simplifies to give $\int 8\sin^2\theta\, d\theta$ including $d\theta$
	2nd B1	Writes down a correct equation involving $x = 3$ leading to $\theta = \frac{\pi}{3}$ and no incorrect work seen regarding limits
	Note	Allow 2nd B1 for $x = 4\sin^2\left(\frac{\pi}{3}\right) = 3$ **and** $x = 4\sin^2 0 = 0$
	Note	Allow 2nd B1 for $\theta = \sin^{-1}\left(\sqrt{\frac{x}{4}}\right)$ followed by $x = 3,\ \theta = \frac{\pi}{3};\ x = 0,\ \theta = 0$

(ii)(b)	M1	Writes down a correct equation involving $\cos 2\theta$ and $\sin^2\theta$ **E.g.:** $\cos 2\theta = 1 - 2\sin^2\theta$ or $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ or $K\sin^2\theta = K\left(\frac{1-\cos 2\theta}{2}\right)$ and *applies* it to their integral. **Note:** Allow M1 for a correctly stated formula (via an incorrect rearrangement) being applied to their integral.
	M1	Integrates to give an expression of the form $\pm\alpha\theta \pm \beta\sin 2\theta$ or $k(\pm\alpha\theta \pm \beta\sin 2\theta)$, $\alpha \neq 0,\ \beta \neq 0$ (can be simplified or un-simplified).
	1st A1	Integrating $\sin^2\theta$ to give $\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$, un-simplified or simplified. Correct solution only. Can be implied by $k\sin^2\theta$ giving $\frac{k}{2}\theta - \frac{k}{4}\sin 2\theta$ or $\frac{k}{4}(2\theta - \sin 2\theta)$ un-simplified or simplified.
	2nd A1	A *correct solution in part (ii)* leading to a "two term" exact answer of e.g. $\frac{4}{3}\pi - \sqrt{3}$ or $\frac{8}{6}\pi - \sqrt{3}$ or $\frac{4}{3}\pi - \frac{2\sqrt{3}}{2}$ or $\frac{1}{3}(4\pi - 3\sqrt{3})$
	Note	A decimal answer of 2.456739397... (without a correct **exact** answer) is A0.
	Note	Candidates can work in terms of λ (note that λ is not given in (ii)) and gain the 1st three marks (i.e. M1M1A1) in part (b).
	Note	If they incorrectly obtain $\int_0^{\frac{\pi}{3}} 8\sin^2\theta\, d\theta$ in part (i)(a) (or correctly guess that $\lambda = 8$) then the final A1 is available for a correct solution in part (ii)(b).

Q4.

Question Number	Scheme		Marks
(a)	$\{x = u^2 \Rightarrow\}\ \frac{dx}{du} = 2u$ or $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$		B1
	$\left\{\int \frac{1}{x(2\sqrt{x}-1)}\,dx\right\} = \int \frac{1}{u^2(2u-1)}\,2u\,du$		M1
	$= \int \frac{2}{u(2u-1)}\,du$		A1 * cso
			[3]
(b)	$\frac{2}{u(2u-1)} \equiv \frac{A}{u} + \frac{B}{(2u-1)} \Rightarrow 2 \equiv A(2u-1) + Bu$		
	$u = 0 \Rightarrow 2 = -A \Rightarrow A = -2$	See notes	M1 A1
	$u = \frac{1}{2} \Rightarrow 2 = \frac{1}{2}B \Rightarrow B = 4$		
	So $\int \frac{2}{u(2u-1)}\,du = \int \frac{-2}{u} + \frac{4}{(2u-1)}\,du$	Integrates $\frac{M}{u} + \frac{N}{(2u-1)}$, $M \neq 0$, $N \neq 0$ to obtain any one of $\pm\lambda\ln u$ or $\pm\mu\ln(2u-1)$	M1
	$= -2\ln u + 2\ln(2u-1)$	At least one term correctly followed through	A1 ft
		$-2\ln u + 2\ln(2u-1)$.	A1 cao
	So, $[-2\ln u + 2\ln(2u-1)]_1^3$		
	$= (-2\ln 3 + 2\ln(2(3)-1)) - (-2\ln 1 + 2\ln(2(1)-1))$	Applies limits of 3 and 1 in u or 9 and 1 in x in their integrated function and subtracts the correct way round.	M1
	$= -2\ln 3 + 2\ln 5 - (0)$		
	$= 2\ln\left(\frac{5}{3}\right)$	$2\ln\left(\frac{5}{3}\right)$	A1 cso cao
			[7]
			10

Q5.

Question	Scheme for Substitution		Marks	AOs
	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,dx$ Award for • Using a valid substitution $u =$..., changing the terms to u's • integrating and using appropriate limts .		M1	3.1a
	Substitution $u=\sqrt{x+2} \Rightarrow \frac{dx}{du}=2u$ oe	Substitution $u=x+2 \Rightarrow \frac{dx}{du}=1$ oe	B1	1.1b
	$\int 2x\sqrt{x+2}\,dx$ $=\int A(u^2 \pm 2)u^2 du$	$\int 2x\sqrt{x+2}\,dx$ $=\int A(u \pm 2)\sqrt{u}\,du$	M1	1.1b
	$=Pu^5 \pm Qu^3$	$=Su^{\frac{5}{2}} \pm Tu^{\frac{3}{2}}$	dM1	2.1
	$=\frac{4}{5}u^5-\frac{8}{3}u^3$	$=\frac{4}{5}u^{\frac{5}{2}}-\frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$=\frac{32}{15}(2+\sqrt{2})$ *		A1*	2.1
			(7)	
				(7 marks)

Q6.

Question Number	Scheme		Marks
(a)	$A=\int_0^3 \sqrt{(3-x)(x+1)}\,dx$, $x=1+2\sin\theta$		
	$\frac{dx}{d\theta}=2\cos\theta$	$\frac{dx}{d\theta}=2\cos\theta$ or $2\cos\theta$ used correctly in their working. Can be implied.	B1
	$\left\{\int\sqrt{(3-x)(x+1)}\,dx \text{ or } \int\sqrt{(3+2x-x^2)}\,dx\right\}$		
	$=\int\sqrt{(3-(1+2\sin\theta))((1+2\sin\theta)+1)}\;2\cos\theta\,\{d\theta\}$	Substitutes for both x and dx, where $dx\neq\lambda d\theta$. Ignore $d\theta$	M1
	$=\int\sqrt{(2-2\sin\theta)(2+2\sin\theta)}\;2\cos\theta\,\{d\theta\}$ $=\int\sqrt{(4-4\sin^2\theta)}\;2\cos\theta\,\{d\theta\}$		
	$=\int\sqrt{(4-4(1-\cos^2\theta))}\;2\cos\theta\,\{d\theta\}$ or $\int\sqrt{4\cos^2\theta}\;2\cos\theta\,\{d\theta\}$	Applies $\cos^2\theta=1-\sin^2\theta$ see notes	M1
	$=4\int\cos^2\theta\,d\theta,\ \{k=4\}$	$4\int\cos^2\theta d\theta$ or $\int 4\cos^2\theta d\theta$ Note: $d\theta$ is required here.	A1
	$0=1+2\sin\theta$ or $-1=2\sin\theta$ or $\sin\theta=-\frac{1}{2}\Rightarrow\underline{\theta=-\frac{\pi}{6}}$ and $3=1+2\sin\theta$ or $2=2\sin\theta$ or $\sin\theta=1\Rightarrow\underline{\theta=\frac{\pi}{2}}$	See notes	B1
			[5]
(b)	$\left\{k\int\cos^2\theta\,\{d\theta\}\right\}=\{k\}\int\left(\frac{1+\cos 2\theta}{2}\right)\{d\theta\}$	Applies $\cos 2\theta=2\cos^2\theta-1$ to their integral	M1
	$=\{k\}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)$	Integrates to give $\pm\alpha\theta\pm\beta\sin 2\theta,\ \alpha\neq 0,\beta\neq 0$ or $k(\pm\alpha\theta\pm\beta\sin 2\theta)$	M1 (A1 on ePEN)
	$\left\{\text{So } 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta d\theta=\left[2\theta+\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}$ $=\left(2\left(\frac{\pi}{2}\right)+\sin\left(\frac{2\pi}{2}\right)\right)-\left(2\left(-\frac{\pi}{6}\right)+\sin\left(-\frac{2\pi}{6}\right)\right)$		
	$\left\{=(\pi)-\left(-\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)\right\}=\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$	$\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$ or $\frac{1}{6}(8\pi+3\sqrt{3})$	A1 cao cso
			[3] 8

Q7.

Question Number	Scheme	Notes	Marks
	x: 0, 0.2, 0.4, 0.6, 0.8, 1; y: 2, ***1.8625426...***, 1.71830, 1.56981, 1.41994, 1.27165	$y=\dfrac{6}{(2+e^x)}$	
(a)	$\{\text{At } x=0.2,\}\ y=1.86254$ (5 dp)	1.86254	B1 cao
	Note: Look for this value on the given table or in their working.		[1]
(b)	$\frac{1}{2}(0.2)\left[2+1.27165+2(\text{their } 1.86254+1.71830+1.56981+1.41994)\right]$	Outside brackets $\frac{1}{2}\times(0.2)$ or $\frac{1}{10}$ or $\frac{1}{2}\times\frac{1}{5}$	B1 o.e.
		For structure of $[\ldots\ldots]$	M1
	$\left\{=\frac{1}{10}(16.41283)\right\}=1.641283=1.6413$ (4 dp)	anything that rounds to 1.6413	A1
			[3]
(c)	$\{u=e^x \text{ or } x=\ln u\Rightarrow\}$		
	$\frac{du}{dx}=e^x$ or $\frac{du}{dx}=u$ or $\frac{dx}{du}=\frac{1}{u}$ or $du=u\,dx$ etc., and $\int\frac{6}{(e^x+2)}dx=\int\frac{6}{(u+2)u}du$	**See notes**	B1 *
	$\{x=0\}\Rightarrow a=e^0\Rightarrow a=1$ $\{x=1\}\Rightarrow b=e^1\Rightarrow b=e$	$a=1$ and $b=e$ or $b=e^1$ or evidence of $0\to1$ and $1\to e$	B1
	NOTE: 1st B1 mark CANNOT be recovered for work in part (d) **NOTE: 2nd B1 mark CAN be recovered for work in part (d)**		[2]
(d) **Way 1**	$\frac{6}{u(u+2)}\equiv\frac{A}{u}+\frac{B}{(u+2)}$ $\Rightarrow 6\equiv A(u+2)+Bu$	Writing $\frac{6}{u(u+2)}\equiv\frac{A}{u}+\frac{B}{(u+2)}$, o.e. or $\frac{1}{u(u+2)}\equiv\frac{P}{u}+\frac{Q}{(u+2)}$, o.e., and a complete method for finding the value of at least one of **their** A or **their** B (or **their** P or **their** Q)	M1
	$u=0\Rightarrow A=3$ $u=-2\Rightarrow B=-3$	**Both their** $A=3$ **and their** $B=-3$. (Or **their** $P=\frac{1}{2}$ **and their** $Q=-\frac{1}{2}$ with the factor of 6 in front of the integral sign)	A1
	$\int\frac{6}{u(u+2)}du=\int\left(\frac{3}{u}-\frac{3}{(u+2)}\right)du$	Integrates $\frac{M}{u}\pm\frac{N}{u\pm k}$, $M,N,k\neq0$; (i.e. ***a two term partial fraction***) to obtain either $\pm\lambda\ln(\alpha u)$ or $\pm\mu\ln(\beta(u\pm k))$; $\lambda,\mu,\alpha,\beta\neq0$	M1
	$=3\ln u-3\ln(u+2)$ or $=3\ln2u-3\ln(2u+4)$	Integration of both terms is **correctly followed through** from **their** M and from **their** N.	A1 ft
	$\left\{\text{So }\left[3\ln u-3\ln(u+2)\right]_1^e\right\}$ $=\left(3\ln(e)-3\ln(e+2)\right)-\left(3\ln1-3\ln3\right)$ [Note: A proper consideration of the limit of $u=1$ is required for this mark]	**dependent on the 2nd M mark** Applies limits of e and 1 (or their b and their a, where $b>0, b\neq1, a>0$) in u or applies limits of 1 and 0 in x and subtracts the correct way round.	dM1
	$=3-3\ln(e+2)+3\ln3$ or $3(1-\ln(e+2)+\ln3)$ or $3+3\ln\left(\frac{3}{e+2}\right)$ or $3\ln\left(\frac{e}{e+2}\right)-3\ln\left(\frac{1}{3}\right)$ or $3-3\ln\left(\frac{e+2}{3}\right)$ or $3\ln\left(\frac{3e}{e+2}\right)$ or $\ln\left(\frac{27e^3}{(e+2)^3}\right)$	**see notes**	A1 cso
	Note: Allow e^1 in place of e for the final A1 mark.		[6]
	Note: Give final A0 for $3-3\ln e+2+3\ln3$ (i.e. bracketing error) unless recovered.		12
	Note: Give final A0 for $3-3\ln(e+2)+3\ln3-3\ln1$, where $3\ln1$ has not been simplified to 0		
	Note: Give final A0 for $3\ln e-3\ln(e+2)+3\ln3$, where $3\ln e$ has not been simplified to 3		

Q8.

Question Number	Scheme		Marks
(a)	0.73508		B1 cao [1]
(b)	Area $= \frac{1}{2}\times\frac{\pi}{8};\times\left[0+2(\text{their } 0.73508+1.17157+1.02280)+0\right]$		B1 M1
	$= \frac{\pi}{16}\times 5.8589... = 1.150392325... = 1.1504$ (4 dp)	awrt 1.1504	A1 [3]
(c)	$\{u=1+\cos x\} \Rightarrow \frac{du}{dx} = -\sin x$		B1
	$\left\{\int\frac{2\sin 2x}{(1+\cos x)}dx=\right\}\int\frac{2(2\sin x\cos x)}{(1+\cos x)}dx$	$\sin 2x = 2\sin x\cos x$	B1
	$=\int\frac{4(u-1)}{u}(-1)du \quad \left\{=4\int\frac{(1-u)}{u}du\right\}$		M1
	$=4\int\left(\frac{1}{u}-1\right)du = 4(\ln u - u)+c$		dM1
	$=4\ln(1+\cos x)-4(1+\cos x)+c \quad = 4\ln(1+\cos x)-4\cos x+k$	AG	A1 cso [5]
(d)	$=\left[4\ln\left(1+\cos\frac{\pi}{2}\right)-4\cos\frac{\pi}{2}\right]-\left[4\ln(1+\cos 0)-4\cos 0\right]$	Applying limits $x=\frac{\pi}{2}$ and $x=0$ either way round.	M1
	$=[4\ln 1-0]-[4\ln 2-4]$		
	$=4-4\ln 2 \ \{=1.227411278...\}$	$\pm4(1-\ln 2)$ or $\pm(4-4\ln 2)$ or awrt ±1.2, *however* found.	A1
	Error $=\lvert(4-4\ln 2)-1.1504...\rvert$ $=0.0770112776... = 0.077$ (2sf)	awrt ±0.077 or awrt $\pm6.3(\%)$	A1 cso [3]
			12

Q9.

Question Number	Scheme		Marks
(a)	1.0981		B1 cao [1]
(b)	Area $\approx \frac{1}{2} \times 1 ; \times \left\{0.5 + 2(0.8284 + \text{their } 1.0981) + 1.3333\right\}$		B1; M1
	$= \frac{1}{2} \times 5.6863 = 2.84315 = 2.843$ (3 dp)	2.843 or awrt 2.843	A1 [3]
(c)	$\left\{u = 1 + \sqrt{x}\right\} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{dx}{du} = 2(u-1)$		B1
	$\left\{\int \frac{x}{1+\sqrt{x}}\, dx =\right\} \int \frac{(u-1)^2}{u} \cdot 2(u-1)\, du$	$\int \frac{(u-1)^2}{u} \ldots$	M1
		$\int \frac{(u-1)^2}{u} \cdot 2(u-1)$	A1
	$= 2\int \frac{(u-1)^3}{u}\, du = \{2\}\int \frac{(u^3 - 3u^2 + 3u - 1)}{u}\, du$	Expands to give a "four term" cubic in u. Eg: $\pm Au^3 \pm Bu^2 \pm Cu \pm D$	M1
	$= \{2\}\int \left(u^2 - 3u + 3 - \frac{1}{u}\right) du$	An attempt to divide at least three terms in *their cubic* by u. See notes.	M1
	$= \{2\}\left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$	$\int \frac{(u-1)^3}{u} \to \left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$	A1
	Area(R) $= \left[\frac{2u^3}{3} - 3u^2 + 6u - 2\ln u\right]_2^3$		
	$= \left(\frac{2(3)^3}{3} - 3(3)^2 + 6(3) - 2\ln 3\right) - \left(\frac{2(2)^3}{3} - 3(2)^2 + 6(2) - 2\ln 2\right)$	Applies limits of 3 and 2 in u or 4 and 1 in x and subtracts either way round.	M1
	$= \frac{11}{3} + 2\ln 2 - 2\ln 3$ or $\frac{11}{3} + 2\ln\left(\frac{2}{3}\right)$ or $\frac{11}{3} - \ln\left(\frac{9}{4}\right)$ etc	Correct exact answer or equivalent	A1
			[8] 12

Q10.

Question Number	Scheme		Marks
(a)	1.0981		B1 cao
			[1]
(b)	Area $\approx \frac{1}{2} \times 1$; $\times \left[0.5 + 2(0.8284 + \text{their } 1.0981) + 1.3333\right]$		B1; M1
	$= \frac{1}{2} \times 5.6863 = 2.84315 = 2.843$ (3 dp)	2.843 or awrt 2.843	A1
			[3]
(c)	$\left\{u = 1 + \sqrt{x}\right\} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\frac{dx}{du} = 2(u-1)$		B1
	$\left\{\int \frac{x}{1+\sqrt{x}}\,dx = \right\} \int \frac{(u-1)^2}{u} \cdot 2(u-1)\,du$	$\int \frac{(u-1)^2}{u} \ldots$	M1
		$\int \frac{(u-1)^2}{u} \cdot 2(u-1)$	A1
	$= 2\int \frac{(u-1)^3}{u}\,du = \{2\} \int \frac{(u^3 - 3u^2 + 3u - 1)}{u}\,du$	Expands to give a "four term" cubic in u. Eg: $\pm Au^3 \pm Bu^2 \pm Cu \pm D$	M1
	$= \{2\} \int \left(u^2 - 3u + 3 - \frac{1}{u}\right) du$	An attempt to divide at least three terms in *their cubic* by u. See notes.	M1
	$= \{2\} \left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$	$\int \frac{(u-1)^3}{u} \to \left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$	A1
	$\text{Area}(R) = \left[\frac{2u^3}{3} - 3u^2 + 6u - 2\ln u\right]_2^3$		
	$= \left(\frac{2(3)^3}{3} - 3(3)^2 + 6(3) - 2\ln 3\right) - \left(\frac{2(2)^3}{3} - 3(2)^2 + 6(2) - 2\ln 2\right)$	Applies limits of 3 and 2 in u or 4 and 1 in x and subtracts either way round.	M1
	$= \frac{11}{3} + 2\ln 2 - 2\ln 3$ or $\frac{11}{3} + 2\ln\left(\frac{2}{3}\right)$ or $\frac{11}{3} - \ln\left(\frac{9}{4}\right)$, etc	Correct exact answer or equivalent.	A1
			[8]
			12