

Question 1(Jun 2005, Q2)

Worked Solution

Evaluate $\int_0^{\pi/2} x \cos x \, dx$, **giving your answer in an exact form.**

Use integration by parts: $u = x$, $\frac{dv}{dx} = \cos x$.

Then $\frac{du}{dx} = 1$, $v = \sin x$.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x (+c)$$

Apply limits:

$$[x \sin x + \cos x]_0^{\pi/2} = \left(\frac{\pi}{2} \cdot 1 + 0\right) - (0 + 1) = \frac{\pi}{2} - 1$$

$$\int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1$$

Question 2

(Jan 2006, Q4)

Worked Solution

(i) Use integration by parts to find $\int x \sec^2 x \, dx$.

Let $u = x$, $\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$.

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + c$$

$$\int x \sec^2 x \, dx = x \tan x + \ln |\cos x| + c$$

(ii) Hence find $\int x \tan^2 x \, dx$.

Use $\tan^2 x = \sec^2 x - 1$:

$$\int x \tan^2 x \, dx = \int x \sec^2 x \, dx - \int x \, dx$$

$$= x \tan x + \ln |\cos x| - \frac{1}{2}x^2 + c$$

$$\int x \tan^2 x \, dx = x \tan x + \ln |\cos x| - \frac{1}{2}x^2 + c$$

Question 3

(Jan 2007, Q2)

Worked Solution

Find the exact value of $\int_1^2 x \ln x \, dx$.

Use integration by parts: $u = \ln x$, $\frac{dv}{dx} = x$.

Then $\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^2}{2}$.

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

Apply limits 1 to 2:

$$\left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_1^2 = (2 \ln 2 - 1) - \left(0 - \frac{1}{4} \right) = 2 \ln 2 - 1 + \frac{1}{4}$$

$$\int_1^2 x \ln x \, dx = 2 \ln 2 - \frac{3}{4}$$

Question 4

(Jun 2007, Q2)

Worked Solution

Find the exact value of $\int_0^1 x^2 e^x dx$.

First application: $u = x^2$, $\frac{dv}{dx} = e^x \Rightarrow v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

Second application: $u = x$, $\frac{dv}{dx} = e^x \Rightarrow v = e^x$.

$$\int x e^x dx = x e^x - e^x + c$$

Combining:

$$\int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x) = (x^2 - 2x + 2)e^x + c$$

Apply limits:

$$\left[(x^2 - 2x + 2)e^x \right]_0^1 = (1 - 2 + 2)e - (0 - 0 + 2)e^0 = e - 2$$

$$\int_0^1 x^2 e^x dx = e - 2$$

Question 5

(Jan 2010, Q8)

Worked Solution

(i) State the derivative of $e^{\cos x}$.

Using the chain rule:

$$\frac{d}{dx}(e^{\cos x}) = -\sin x e^{\cos x}$$

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(ii) Hence use integration by parts to find the exact value of $\int_0^{\pi/2} \cos x \sin x e^{\cos x} dx$.

From (i), $\int \sin x e^{\cos x} dx = -e^{\cos x} + c$.

Use IBP: $u = \cos x$, $\frac{dv}{dx} = \sin x e^{\cos x}$, so $v = -e^{\cos x}$.

$$\frac{du}{dx} = -\sin x.$$

$$\begin{aligned} \int \cos x \sin x e^{\cos x} dx &= -\cos x e^{\cos x} - \int (-e^{\cos x})(-\sin x) dx \\ &= -\cos x e^{\cos x} - \int \sin x e^{\cos x} dx = -\cos x e^{\cos x} + e^{\cos x} + c = e^{\cos x}(1 - \cos x) + c \end{aligned}$$

Apply limits 0 to $\pi/2$:

$$\left[e^{\cos x}(1 - \cos x) \right]_0^{\pi/2} = (e^0 \cdot 1) - (e^1 \cdot 0) = 1$$

$$\int_0^{\pi/2} \cos x \sin x e^{\cos x} dx = 1$$

Question 6

(Jun 2010, Q9i)

Worked Solution

Find $\int (x + \cos 2x)^2 dx$.

Expand:

$$(x + \cos 2x)^2 = x^2 + 2x \cos 2x + \cos^2 2x$$

Integrate x^2 : $\frac{x^3}{3}$.

Integrate $2x \cos 2x$: IBP with $u = 2x$, $\frac{dv}{dx} = \cos 2x \Rightarrow v = \frac{\sin 2x}{2}$.

$$\int 2x \cos 2x dx = x \sin 2x - \int \sin 2x dx = x \sin 2x + \frac{\cos 2x}{2}$$

Integrate $\cos^2 2x$: Using $\cos^2 2x = \frac{1 + \cos 4x}{2}$:

$$\int \cos^2 2x dx = \frac{x}{2} + \frac{\sin 4x}{8}$$

Combining:

$$\int (x + \cos 2x)^2 dx = \frac{x^3}{3} + x \sin 2x + \frac{\cos 2x}{2} + \frac{x}{2} + \frac{\sin 4x}{8} + c$$

$$\int (x + \cos 2x)^2 dx = \frac{x^3}{3} + x \sin 2x + \frac{\cos 2x}{2} + \frac{x}{2} + \frac{\sin 4x}{8} + c$$

Question 7

(Jun 2013, Q2)

Worked Solution

Find $\int x^8 \ln(3x) dx$.Use IBP: $u = \ln(3x)$, $\frac{dv}{dx} = x^8$.

$$\frac{du}{dx} = \frac{1}{x}, v = \frac{x^9}{9}.$$

$$\begin{aligned}\int x^8 \ln(3x) dx &= \frac{x^9}{9} \ln(3x) - \int \frac{x^9}{9} \cdot \frac{1}{x} dx = \frac{x^9}{9} \ln(3x) - \int \frac{x^8}{9} dx \\ &= \frac{x^9}{9} \ln(3x) - \frac{x^9}{81} + c\end{aligned}$$

$$\int x^8 \ln(3x) dx = \frac{x^9}{9} \ln(3x) - \frac{x^9}{81} + c$$

Question 8

(Jun 2014, Q8)

Worked Solution

(i) Use division to show that $\frac{t^3}{t+2} \equiv t^2 - 2t + 4 - \frac{8}{t+2}$.

Performing polynomial long division of $t^3 \div (t+2)$:

$t^3 \div t = t^2$. Then $t^2(t+2) = t^3 + 2t^2$. Remainder: $-2t^2$.

$-2t^2 \div t = -2t$. Then $-2t(t+2) = -2t^2 - 4t$. Remainder: $4t$.

$4t \div t = 4$. Then $4(t+2) = 4t + 8$. Remainder: -8 .

$$\frac{t^3}{t+2} = t^2 - 2t + 4 - \frac{8}{t+2} \quad \checkmark$$

(ii) Find $\int_1^2 6t^2 \ln(t+2) dt$, giving your answer in the form $A + B \ln 3 + C \ln 4$.

Use IBP: $u = \ln(t+2)$, $\frac{dv}{dt} = 6t^2 \Rightarrow v = 2t^3$.

$$\int 6t^2 \ln(t+2) dt = 2t^3 \ln(t+2) - \int \frac{2t^3}{t+2} dt$$

From part (i): $\frac{t^3}{t+2} = t^2 - 2t + 4 - \frac{8}{t+2}$, so:

$$\int \frac{2t^3}{t+2} dt = 2 \int \left(t^2 - 2t + 4 - \frac{8}{t+2} \right) dt = \frac{2t^3}{3} - 2t^2 + 8t - 16 \ln(t+2) + c$$

Therefore:

$$\int_1^2 6t^2 \ln(t+2) dt = \left[2t^3 \ln(t+2) - \frac{2t^3}{3} + 2t^2 - 8t + 16 \ln(t+2) \right]_1^2$$

At $t = 2$: $16 \ln 4 - \frac{16}{3} + 8 - 16 + 16 \ln 4 = 32 \ln 4 - \frac{16}{3} - 8$.

At $t = 1$: $2 \ln 3 - \frac{2}{3} + 2 - 8 + 16 \ln 3 = 18 \ln 3 - \frac{2}{3} - 6$.

Subtracting:

$$\begin{aligned} &= 32 \ln 4 - 18 \ln 3 - \frac{16}{3} + \frac{2}{3} - 8 + 6 = 32 \ln 4 - 18 \ln 3 - \frac{14}{3} - 2 \\ &= -\frac{20}{3} + 32 \ln 4 - 18 \ln 3 \end{aligned}$$

Checking against the required form $A + B \ln 3 + C \ln 4$:

$$\int_1^2 6t^2 \ln(t+2) dt = -\frac{20}{3} - 18 \ln 3 + 32 \ln 4$$

So $A = -\frac{20}{3}$, $B = -18$, $C = 32$.

Note: This can also be written as $-\frac{20}{3} + \ln \frac{4^{32}}{3^{18}}$.

Question 9

(Jun 2016, Q4)

Worked Solution

Find the exact value of $\int_1^8 \frac{1}{\sqrt[3]{x}} \ln x \, dx$, giving your answer in the form $A \ln 2 + B$.

Rewrite: $\frac{1}{\sqrt[3]{x}} = x^{-1/3}$.

Use IBP: $u = \ln x$, $\frac{dv}{dx} = x^{-1/3}$.

$\frac{du}{dx} = \frac{1}{x}$, $v = \frac{x^{2/3}}{2/3} = \frac{3}{2}x^{2/3}$.

$$\begin{aligned} \int x^{-1/3} \ln x \, dx &= \frac{3}{2}x^{2/3} \ln x - \int \frac{3}{2}x^{2/3} \cdot \frac{1}{x} \, dx = \frac{3}{2}x^{2/3} \ln x - \frac{3}{2} \int x^{-1/3} \, dx \\ &= \frac{3}{2}x^{2/3} \ln x - \frac{3}{2} \cdot \frac{x^{2/3}}{2/3} = \frac{3}{2}x^{2/3} \ln x - \frac{9}{4}x^{2/3} + c \end{aligned}$$

Apply limits 1 to 8. Note $8^{2/3} = 4$, $1^{2/3} = 1$, $\ln 8 = 3 \ln 2$, $\ln 1 = 0$:

$$\begin{aligned} \left[\frac{3}{2}x^{2/3} \ln x - \frac{9}{4}x^{2/3} \right]_1^8 &= \left(\frac{3}{2} \cdot 4 \cdot 3 \ln 2 - \frac{9}{4} \cdot 4 \right) - \left(0 - \frac{9}{4} \right) \\ &= 18 \ln 2 - 9 + \frac{9}{4} = 18 \ln 2 - \frac{27}{4} \end{aligned}$$

$$\int_1^8 \frac{1}{\sqrt[3]{x}} \ln x \, dx = 18 \ln 2 - \frac{27}{4}$$

So $A = 18$, $B = -\frac{27}{4}$.

End of Worked Solutions