Question 1

Worked Solution

(a) Use integration by parts to find $\int x \sin 3x \, \mathrm{d}x$**.**

Let $u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\frac{\cos 3x}{3}$.

$$\int x \sin 3x \, \mathrm{d}x = -\frac{x \cos 3x}{3} + \int \frac{\cos 3x}{3} \, \mathrm{d}x = -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + c$$

$$\int x \sin 3x \, \mathrm{d}x = -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + c$$

(b) Using your answer to (a), find $\int x^2 \cos 3x \, \mathrm{d}x$**.**

Let $u = x^2$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 3x \Rightarrow v = \frac{\sin 3x}{3}$.

$$\int x^2 \cos 3x \, \mathrm{d}x = \frac{x^2 \sin 3x}{3} - \frac{2}{3} \int x \sin 3x \, \mathrm{d}x$$

Substitute part (a):

$$= \frac{x^2 \sin 3x}{3} - \frac{2}{3}\left(-\frac{x \cos 3x}{3} + \frac{\sin 3x}{9}\right) + c$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + c$$

$$\int x^2 \cos 3x \, \mathrm{d}x = \frac{x^2 \sin 3x}{3} + \frac{2x \cos 3x}{9} - \frac{2 \sin 3x}{27} + c$$

Question 2

Worked Solution

Use integration to find the exact value of $\int_0^{\pi/2} x \sin 2x \, \mathrm{d}x$**.**

Let $u = x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \Rightarrow v = -\dfrac{\cos 2x}{2}$.

$$\int x \sin 2x \, \mathrm{d}x = -\frac{x \cos 2x}{2} + \int \frac{\cos 2x}{2} \, \mathrm{d}x = -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} + c$$

Apply limits:

$$\left[-\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} \right]_0^{\pi/2} = \left(-\frac{\pi}{2} \cdot \frac{\cos \pi}{2} + \frac{\sin \pi}{4} \right) - (0 + 0)$$

$$= \left(\frac{\pi}{4} + 0 \right) - 0 = \frac{\pi}{4}$$

$$\int_0^{\pi/2} x \sin 2x \, \mathrm{d}x = \frac{\pi}{4}$$

Question 3

Worked Solution

(a) Use integration by parts to find $\int xe^x \, dx$.

Let $u = x$, $\frac{dv}{dx} = e^x \Rightarrow v = e^x$.

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + c = (x-1)e^x + c$$

$$\int xe^x \, dx = (x-1)e^x + c$$

(b) Hence find $\int x^2 e^x \, dx$.

Let $u = x^2$, $\frac{dv}{dx} = e^x \Rightarrow v = e^x$.

$$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx = x^2 e^x - 2(x-1)e^x + c$$
$$= e^x(x^2 - 2x + 2) + c$$

$$\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + c$$

Question 4

Worked Solution

$f(\theta) = 4\cos^2\theta - 3\sin^2\theta$.

(a) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2}\cos 2\theta$**.**

Use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$$f(\theta) = 4 \cdot \frac{1+\cos 2\theta}{2} - 3 \cdot \frac{1-\cos 2\theta}{2} = 2(1+\cos 2\theta) - \frac{3}{2}(1-\cos 2\theta)$$

$$= 2 + 2\cos 2\theta - \frac{3}{2} + \frac{3}{2}\cos 2\theta = \frac{1}{2} + \frac{7}{2}\cos 2\theta \quad \checkmark$$

(b) Hence find the exact value of $\int_0^{\pi/2} \theta\, f(\theta)\, d\theta$**.**

$$\int_0^{\pi/2} \theta\, f(\theta)\, d\theta = \int_0^{\pi/2} \theta \left(\frac{1}{2} + \frac{7}{2}\cos 2\theta\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \theta\, d\theta + \frac{7}{2}\int_0^{\pi/2} \theta \cos 2\theta\, d\theta$$

First integral: $\frac{1}{2}\left[\frac{\theta^2}{2}\right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{16}$.

Second integral (IBP: $u = \theta$, $v' = \cos 2\theta \Rightarrow v = \frac{\sin 2\theta}{2}$):

$$\int_0^{\pi/2} \theta \cos 2\theta\, d\theta = \left[\frac{\theta \sin 2\theta}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta = 0 + \left[\frac{\cos 2\theta}{4}\right]_0^{\pi/2} = \frac{-1-1}{4} = -\frac{1}{2}$$

Combining:

$$= \frac{\pi^2}{16} + \frac{7}{2} \cdot \left(-\frac{1}{2}\right) = \frac{\pi^2}{16} - \frac{7}{4}$$

$$\int_0^{\pi/2} \theta\, f(\theta)\, d\theta = \frac{\pi^2}{16} - \frac{7}{4}$$

Question 5

Worked Solution

Curve C: $y = x \ln x$, $x > 0$. Normal l to C at $P(e, e)$. Region R bounded by C, l and x-axis.

Show that the exact area of R is Ae^2+B where A and B are rational numbers.

Step 1: Find the gradient of the tangent at $P(e, e)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

At $x = e$: $m_T = \ln e + 1 = 2$. So $m_N = -\frac{1}{2}$.

Step 2: Equation of normal l.

$$y - e = -\tfrac{1}{2}(x - e) \implies y = -\tfrac{x}{2} + \tfrac{3e}{2}$$

l meets x-axis when $y = 0$: $x = 3e$.

Step 3: Area R_1 under curve from $x = 1$ (where $y = 0$) to $x = e$ (using IBP).

$$\int_1^e x \ln x \, \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right]_1^e = \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

Step 4: Area R_2 of triangle under normal from $x = e$ to $x = 3e$.

$$R_2 = \tfrac{1}{2} \times (3e - e) \times e = \tfrac{1}{2} \times 2e \times e = e^2$$

(Alternatively: $\int_e^{3e} \left(-\frac{x}{2} + \frac{3e}{2}\right) \mathrm{d}x = \left[-\frac{x^2}{4} + \frac{3ex}{2}\right]_e^{3e} = \left(-\frac{9e^2}{4} + \frac{9e^2}{2}\right) - \left(-\frac{e^2}{4} + \frac{3e^2}{2}\right) = e^2$.)

Step 5: Total area.

$$\text{Area}(R) = R_1 + R_2 = \frac{e^2}{4} + \frac{1}{4} + e^2 = \frac{5e^2}{4} + \frac{1}{4}$$

Area $= \dfrac{5}{4}e^2 + \dfrac{1}{4}$, so $A = \dfrac{5}{4}$, $B = \dfrac{1}{4}$.

Question 6

Worked Solution

Curve: $y = 4x - xe^{x/2}$, $x \geq 0$.

(a) Find, in terms of $\ln 2$, the x-coordinate of A (where curve cuts positive x-axis).

$y = 0$: $4x - xe^{x/2} = 0 \Rightarrow x(4 - e^{x/2}) = 0$.

Since $x > 0$: $e^{x/2} = 4 \Rightarrow \dfrac{x}{2} = \ln 4 \Rightarrow x = 2\ln 4 = 4\ln 2$.

$x_A = 4\ln 2$

(b) Find $\int xe^{x/2}\,\mathrm{d}x$.

IBP: $u = x$, $v' = e^{x/2} \Rightarrow v = 2e^{x/2}$.

$$\int xe^{x/2}\,\mathrm{d}x = 2xe^{x/2} - 2\int e^{x/2}\,\mathrm{d}x = 2xe^{x/2} - 4e^{x/2} + c$$

$\int xe^{x/2}\,\mathrm{d}x = 2xe^{x/2} - 4e^{x/2} + c$

(c) Find the exact area of R in terms of $\ln 2$.

$$\text{Area} = \int_0^{4\ln 2} (4x - xe^{x/2})\,\mathrm{d}x = \left[2x^2 - 2xe^{x/2} + 4e^{x/2}\right]_0^{4\ln 2}$$

At $x = 4\ln 2$: $e^{x/2} = e^{2\ln 2} = 4$.

$$2(4\ln 2)^2 - 2(4\ln 2)(4) + 4(4) = 32(\ln 2)^2 - 32\ln 2 + 16$$

At $x = 0$: $0 - 0 + 4(1) = 4$.

$$\text{Area} = 32(\ln 2)^2 - 32\ln 2 + 16 - 4 = 32(\ln 2)^2 - 32\ln 2 + 12$$

$\text{Area} = 32(\ln 2)^2 - 32\ln 2 + 12$

Question 7

Worked Solution

(a) Use integration to find $\int \frac{1}{x^3} \ln x \, \mathrm{d}x$**.**

Write as $\int x^{-3} \ln x \, \mathrm{d}x$.

IBP: $u = \ln x$, $v' = x^{-3} \Rightarrow v = -\frac{1}{2x^2}$.

$$\int x^{-3} \ln x \, \mathrm{d}x = -\frac{\ln x}{2x^2} + \int \frac{1}{2x^2} \cdot \frac{1}{x} \, \mathrm{d}x = -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3} \, \mathrm{d}x$$
$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$$

$$\int \frac{\ln x}{x^3} \, \mathrm{d}x = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$$

(b) Hence calculate $\int_1^2 \frac{1}{x^3} \ln x \, \mathrm{d}x$**.**

$$\left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^2 = \left(-\frac{\ln 2}{8} - \frac{1}{16}\right) - \left(0 - \frac{1}{4}\right) = -\frac{\ln 2}{8} + \frac{3}{16}$$

$$\int_1^2 \frac{\ln x}{x^3} \, \mathrm{d}x = \frac{3}{16} - \frac{\ln 2}{8}$$

Question 8

Worked Solution

(a) Find $\int x \cos 2x \, dx$**.**

IBP: $u = x$, $v' = \cos 2x \Rightarrow v = \dfrac{\sin 2x}{2}$.

$$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

$$\int x \cos 2x \, dx = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

(b) Hence, using $\cos 2x = 2\cos^2 x - 1$**, deduce** $\int x \cos^2 x \, dx$**.**

From $\cos 2x = 2\cos^2 x - 1$: $\cos^2 x = \dfrac{\cos 2x + 1}{2}$.

$$\begin{aligned}\int x \cos^2 x \, dx = \int x \cdot \frac{\cos 2x + 1}{2} \, dx &= \frac{1}{2}\int x \cos 2x \, dx + \frac{1}{2}\int x \, dx \\ &= \frac{1}{2}\left(\frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right) + \frac{x^2}{4} + c \\ &= \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^2}{4} + c\end{aligned}$$

$$\int x \cos^2 x \, dx = \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^2}{4} + c$$

Question 9

Worked Solution

(a) Find $\int \sqrt{5-x}\,\mathrm{d}x$**.**

$$\int (5-x)^{1/2}\,\mathrm{d}x = -\frac{2}{3}(5-x)^{3/2} + c$$

$$\int \sqrt{5-x}\,\mathrm{d}x = -\frac{2}{3}(5-x)^{3/2} + c$$

(b)(i) Using integration by parts, find $\int (x-1)\sqrt{5-x}\,\mathrm{d}x$**.**

IBP: $u = x-1$, $v' = \sqrt{5-x} \Rightarrow v = -\frac{2}{3}(5-x)^{3/2}$.

$$\int (x-1)\sqrt{5-x}\,\mathrm{d}x = -\frac{2}{3}(x-1)(5-x)^{3/2} + \frac{2}{3}\int (5-x)^{3/2}\,\mathrm{d}x$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} + \frac{2}{3}\cdot\left(-\frac{2}{5}\right)(5-x)^{5/2} + c$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} - \frac{4}{15}(5-x)^{5/2} + c$$

$$\int (x-1)\sqrt{5-x}\,\mathrm{d}x = -\frac{2}{3}(x-1)(5-x)^{3/2} - \frac{4}{15}(5-x)^{5/2} + c$$

(b)(ii) Hence find $\int_1^5 (x-1)\sqrt{5-x}\,\mathrm{d}x$**.**

$$\left[-\frac{2}{3}(x-1)(5-x)^{3/2} - \frac{4}{15}(5-x)^{5/2}\right]_1^5$$

At $x=5$: both terms $=0$.
At $x=1$: $-\frac{2}{3}(0)(4)^{3/2} - \frac{4}{15}(4)^{5/2} = 0 - \frac{4}{15}(32) = -\frac{128}{15}$.

$$= 0 - \left(-\frac{128}{15}\right) = \frac{128}{15}$$

$$\int_1^5 (x-1)\sqrt{5-x}\,\mathrm{d}x = \frac{128}{15}$$

End of Worked Solutions