Integration By Parts (Sheet 2) Mark Scheme

Q1.

Question Number	Scheme	Marks
(a)	$\int x\sin 3x\,\mathrm{d}x = -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,\{\mathrm{d}x\}$	M1 A1
	$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\,\{+c\}$	A1
		[3]
(b)	$\int x^2\cos 3x\,\mathrm{d}x = \frac{1}{3}x^2\sin 3x - \int \frac{2}{3}x\sin 3x\,\{\mathrm{d}x\}$	M1 A1
	$= \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right)\ \{+c\}$	A1 isw
	$\left\{= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x\ \{+c\}\right\}$ *Ignore subsequent working*	[3]
		6
(a)	**M1:** Use of 'integration by parts' formula $uv - \int vu'$ (whether stated or not stated) in the **correct direction**, where $u = x \to u' = 1$ and $v' = \sin 3x \to v = k\cos 3x$ (seen or implied), where k is a positive or negative constant. (Allow $k = 1$).	
	This means that the candidate must achieve $x(k\cos 3x) - \int (k\cos 3x)$, where k is a consistent constant.	
	If x^2 appears after the integral, this would imply that the candidate is applying integration by parts in the wrong direction, so M0.	
	A1: $-\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,\{\mathrm{d}x\}$. Can be un-simplified. Ignore the $\{\mathrm{d}x\}$.	
	A1: $-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x$ with/without $+c$. Can be un-simplified.	
(b)	**M1:** Use of 'integration by parts' formula $uv - \int vu'$ (whether stated or not stated) in the **correct direction**, where $u = x^2 \to u' = 2x$ or x and $v' = \cos 3x \to v = \lambda\sin 3x$ (seen or implied), where λ is a positive or negative constant. (Allow $\lambda = 1$).	
	This means that the candidate must achieve $x^2(\lambda\sin 3x) - \int 2x(\lambda\sin 3x)$, where $u' = 2x$ or $x^2(\lambda\sin 3x) - \int x(\lambda\sin 3x)$, where $u' = x$.	
	If x^3 appears after the integral, this would imply that the candidate is applying integration by parts in the wrong direction, so M0.	
	A1: $\frac{1}{3}x^2\sin 3x - \int \frac{2}{3}x\sin 3x\,\{\mathrm{d}x\}$. Can be un-simplified. Ignore the $\{\mathrm{d}x\}$.	
	A1: $\frac{1}{3}x^2\sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x\right)$ with/without $+c$, can be un-simplified.	
	You can ignore subsequent working here.	
	Special Case: If the candidate scores the first two marks of M1A1 in part (b), then you can award the final A1 as a follow through for $\frac{1}{3}x^2\sin 3x - \frac{2}{3}$(their follow through part(a) answer).	

Q2.

Question Number	Scheme	Marks
	$\int x\sin 2x\,dx = -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx$	M1 A1 A1
	$= \quad \ldots \quad + \frac{\sin 2x}{4}$	M1
	$[\ \ldots\ ]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$	M1 A1
		[6]

Q3.

Question Number	Scheme	Marks
(a)	$\begin{Bmatrix} u = x & \Rightarrow & \frac{du}{dx} = 1 \\ \frac{dv}{dx} = e^x & \Rightarrow & v = e^x \end{Bmatrix}$	
	$\int xe^x\,dx = xe^x - \int e^x.1\,dx$	M1 A1
	$= xe^x - \int e^x\,dx$	
	$= xe^x - e^x\ (+c)$	A1 (3)
(b)	$\begin{Bmatrix} u = x^2 & \Rightarrow & \frac{du}{dx} = 2x \\ \frac{dv}{dx} = e^x & \Rightarrow & v = e^x \end{Bmatrix}$	
	$\int x^2e^x\,dx = x^2e^x - \int e^x.2x\,dx$	M1 A1
	$= x^2e^x - 2\int xe^x\,dx$	
	$= x^2e^x - 2\left(xe^x - e^x\right) + c$	A1 (3)
		(6 marks)

Q4.

Question Number	Scheme		Marks	
	(a) $f(\theta) = 4\cos^2\theta - 3\sin^2\theta$			
	$= 4\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) - 3\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)$		M1 M1	
	$= \frac{1}{2}+\frac{7}{2}\cos 2\theta$ *	cso	A1	(3)
	(b) $\int \theta\cos 2\theta \, d\theta = \frac{1}{2}\theta\sin 2\theta - \frac{1}{2}\int \sin 2\theta \, d\theta$		M1 A1	
	$= \frac{1}{2}\theta\sin 2\theta + \frac{1}{4}\cos 2\theta$		A1	
	$\int \theta f(\theta) \, d\theta = \frac{1}{4}\theta^2 + \frac{7}{4}\theta\sin 2\theta + \frac{7}{8}\cos 2\theta$		M1 A1	
	$\left[\ \dots\ \right]_0^{\frac{\pi}{2}} = \left[\frac{\pi^2}{16}+0-\frac{7}{8}\right] - \left[0+0+\frac{7}{8}\right]$		M1	
	$= \frac{\pi^2}{16} - \frac{7}{4}$		A1	(7)
				[10]

Q5.

Question	Scheme	Marks	AOs
	$C: y = x\ln x$; l is a normal to C at $P(\mathrm{e}, \mathrm{e})$ Let x_A be the x-coordinate of where l cuts the x-axis		
	$\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$ $\{= 1 + \ln x\}$	M1 A1	2.1 1.1b
	$x = \mathrm{e},\ m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - \mathrm{e} = -\frac{1}{2}(x - \mathrm{e})$ $y = 0 \Rightarrow -\mathrm{e} = -\frac{1}{2}(x-\mathrm{e}) \Rightarrow x = \dots$	M1	3.1a
	l meets x-axis at $x = 3\mathrm{e}$ (allow $x = 2\mathrm{e} + \mathrm{e}\ln\mathrm{e}$)	A1	1.1b
	{Areas:} **either** $\int_1^{\mathrm{e}} x\ln x \, dx = [\ \dots\ ]_1^{\mathrm{e}} = \dots$ **or** $\frac{1}{2}((\text{their } x_A) - \mathrm{e})\mathrm{e}$	M1	2.1
	$\left\{\int x\ln x \, dx = \right\}\ \frac{1}{2}x^2\ln x - \int \frac{1}{x}\cdot\left(\frac{x^2}{2}\right)\{dx\}$	M1	2.1
	$\left\{= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\{dx\}\right\} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$	dM1 A1	1.1b 1.1b
	$\text{Area}(R_1) = \int_1^{\mathrm{e}} x\ln x \, dx = [\ \dots\ ]_1^{\mathrm{e}} = \dots$; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - \mathrm{e})\mathrm{e}$ and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2)$ $\{= \frac{1}{4}\mathrm{e}^2 + \frac{1}{4} + \mathrm{e}^2\}$	M1	3.1a
	$\text{Area}(R) = \frac{5}{4}\mathrm{e}^2 + \frac{1}{4}$	A1	1.1b
		(10)	

Q6.

Question Number	Scheme		Marks
(a)	$y = 4x - xe^{\frac{1}{2}x}$, $x \geqslant 0$ $\left\{y = 0 \Rightarrow 4x - xe^{\frac{1}{2}x} = 0 \Rightarrow x(4 - e^{\frac{1}{2}x}) = 0 \Rightarrow\right\}$		
	$e^{\frac{1}{2}x} = 4 \Rightarrow x_A = 4\ln 2$	Attempts to solve $e^{\frac{1}{2}x} = 4$ giving $x = ...$ in terms of $\pm\lambda \ln \mu$ where $\mu > 0$	M1
		$4\ln 2$ cao (Ignore $x = 0$)	A1
			[2]
(b)	$\left\{\int xe^{\frac{1}{2}x}\,dx\right\} = 2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x}\{dx\}$	$\alpha xe^{\frac{1}{2}x} - \beta\int e^{\frac{1}{2}x}\{dx\}$, $\alpha > 0, \beta > 0$	M1
		$2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x}\{dx\}$, with or without dx	A1 (M1 on ePEN)
	$= 2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\ \{+c\}$	$2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}$ o.e. with or without $+c$	A1
			[3]
(c)	$\left\{\int 4x\,dx\right\} = 2x^2$	$4x \to 2x^2$ or $\frac{4x^2}{2}$ o.e.	B1
	$\left\{\int_0^{4\ln 2}(4x - xe^{\frac{1}{2}x})\,dx\right\} = \left[2x^2 - \left(2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\right)\right]_0^{4\ln 2 \text{ or } \ln 16 \text{ or their limits}}$		
	$= \left(2(4\ln 2)^2 - 2(4\ln 2)e^{\frac{1}{2}(4\ln 2)} + 4e^{\frac{1}{2}(4\ln 2)}\right) - \left(2(0)^2 - 2(0)e^{\frac{1}{2}(0)} + 4e^{\frac{1}{2}(0)}\right)$	See notes	M1
	$= \left(32(\ln 2)^2 - 32(\ln 2) + 16\right) - (4)$ $= 32(\ln 2)^2 - 32(\ln 2) + 12$	$32(\ln 2)^2 - 32(\ln 2) + 12$, see notes	A1
			[3] 8

Q7.

Question Number	Scheme	
(a)	$\int \frac{1}{x^3}\ln x \, dx$, $\left\{ u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}; \quad \frac{dv}{dx} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2} = \frac{-1}{2x^2} \right\}$	
	In the form $\frac{\pm\lambda}{x^2}\ln x \pm \int \mu \frac{1}{x^2}\cdot\frac{1}{x}$	M1
	$= \frac{-1}{2x^2}\ln x - \int \frac{-1}{2x^2}\cdot\frac{1}{x}\,dx$ $\frac{-1}{2x^2}\ln x$ simplified or un-simplified.	A1
	$-\int \frac{-1}{2x^2}\cdot\frac{1}{x}$ simplified or un-simplified.	A1
	$\left\{ = \frac{-1}{2x^2}\ln x + \frac{1}{2}\int \frac{1}{x^3}\,dx \right\}$	
	$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\left(-\frac{1}{2x^2}\right) \{+c\}$ $\pm\int \mu \frac{1}{x^2}\cdot\frac{1}{x} \to \pm\beta x^{-2}$.	dM1
	Correct answer, with/without $+c$	A1
		[5]
(b)	$\left\{\left[-\frac{1}{2x^2}\ln x - \frac{1}{4x^2}\right]_1^2\right\} = \left(-\frac{1}{2(2)^2}\ln 2 - \frac{1}{4(2)^2}\right) - \left(-\frac{1}{2(1)^2}\ln 1 - \frac{1}{4(1)^2}\right)$ Applies limits of 2 and 1 to their part (a) answer and subtracts the correct way round.	M1
	$= \frac{3}{16} - \frac{1}{8}\ln 2$ or $\frac{3}{16} - \ln 2^{\frac{1}{8}}$ or $\frac{1}{16}(3 - 2\ln 2)$, etc, or awrt 0.1 or equivalent.	A1
		[2] 7

Q8.

Question Number	Scheme		Marks
(a)	$\left\{\begin{matrix} u = x & \Rightarrow & \frac{du}{dx} = 1 \\ \frac{dv}{dx} = \cos 2x & \Rightarrow & v = \frac{1}{2}\sin 2x \end{matrix}\right\}$		
	$\text{Int} = \int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x.1\,dx$	***(see note below)*** Use of 'integration by parts' formula in the correct direction. Correct expression.	M1 A1
	$= \frac{1}{2}x\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + c$	$\sin 2x \to -\frac{1}{2}\cos 2x$ or $\sin kx \to -\frac{1}{k}\cos kx$ with $k \neq 1,\ k > 0$	dM1
	$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$	Correct expression with $+c$	A1 **[4]**
(b)	$\int x\cos^2 x\,dx = \int x\left(\frac{\cos 2x + 1}{2}\right)dx$	Substitutes **correctly** for $\cos^2 x$ in the given integral	M1
	$= \frac{1}{2}\int x\cos 2x\,dx + \frac{1}{2}\int x\,dx$		
	$= \underline{\frac{1}{2}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)}; + \frac{1}{2}\int x\,dx$	$\frac{1}{2}$(their answer to (a)); or <u>underlined expression</u>	A1; $\sqrt{}$
	$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \frac{1}{4}x^2\ (+c)$	Completely correct expression with/without $+c$	A1 **[3]**
			7 marks

Notes:

(b)	$\text{Int} = \int x\cos 2x\,dx = \frac{1}{2}x\sin 2x \pm \int \frac{1}{2}\sin 2x.1\,dx$	This is acceptable for M1	M1
	$\left\{\begin{matrix} u = x & \Rightarrow & \frac{du}{dx} = 1 \\ \frac{dv}{dx} = \cos 2x & \Rightarrow & v = \lambda\sin 2x \end{matrix}\right\}$ $\text{Int} = \int x\cos 2x\,dx = \lambda x\sin 2x \pm \int \lambda\sin 2x.1\,dx$	This is also acceptable for M1	M1

Q9.

Question Number	Scheme	Marks
Q (a)	$\int \sqrt{(5-x)}\,\mathrm{d}x = \int (5-x)^{\frac{1}{2}}\,\mathrm{d}x = \dfrac{(5-x)^{\frac{3}{2}}}{-\frac{3}{2}} \quad (+C)$	M1 A1 (2)
	$\left(= -\dfrac{2}{3}(5-x)^{\frac{3}{2}} + C\right)$	
(b)	(i) $\int (x-1)\sqrt{(5-x)}\,\mathrm{d}x = -\dfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} + \dfrac{2}{3}\int (5-x)^{\frac{3}{2}}\,\mathrm{d}x$	M1 A1ft
	$= \quad \ldots \quad + \dfrac{2}{3} \times \dfrac{(5-x)^{\frac{5}{2}}}{-\frac{5}{2}} \quad (+C)$	M1
	$= -\dfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \dfrac{4}{15}(5-x)^{\frac{5}{2}} \quad (+C)$	A1 (4)
	(ii) $\left[-\dfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \dfrac{4}{15}(5-x)^{\frac{5}{2}}\right]_1^5 = (0-0) - \left(0 - \dfrac{4}{15} \times 4^{\frac{5}{2}}\right)$	
	$= \dfrac{128}{15}\left(= 8\dfrac{8}{15} \approx 8.53\right)$ awrt 8.53	M1 A1 (2)
		[8]
	Alternatives for (b) *and* (c)	
	(b) $u^2 = 5 - x \Rightarrow 2u\dfrac{\mathrm{d}u}{\mathrm{d}x} = -1 \quad \left(\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}u} = -2u\right)$	
	$\int (x-1)\sqrt{(5-x)}\,\mathrm{d}x = \int (4-u^2)u\dfrac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u = \int (4-u^2)u(-2u)\,\mathrm{d}u$	M1 A1
	$= \int (2u^4 - 8u^2)\,\mathrm{d}u = \dfrac{2}{5}u^5 - \dfrac{8}{3}u^3 \quad (+C)$	M1
	$= \dfrac{2}{5}(5-x)^{\frac{5}{2}} - \dfrac{8}{3}(5-x)^{\frac{3}{2}} \quad (+C)$	A1
	(c) $x = 1 \Rightarrow u = 2, \quad x = 5 \Rightarrow u = 0$	
	$\left[\dfrac{2}{5}u^5 - \dfrac{8}{3}u^3\right]_2^0 = (0-0) - \left(\dfrac{64}{5} - \dfrac{64}{3}\right)$	M1
	$= \dfrac{128}{15}\left(= 8\dfrac{8}{15} \approx 8.53\right)$ awrt 8.53	A1 (2)