Question 1

(Jan 2006, Q8)

Worked Solution

The curve has parametric equations $x = t - 2\sin t$, $y = 1 - 2\cos t$, $0 \le t \le 2\pi$.

Part (a): Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Set $y = 0$:

$$1 - 2\cos t = 0 \implies \cos t = \tfrac{1}{2}$$

In $[0, 2\pi]$ this gives $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. ✓

$t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$

Part (b): Show that the area of R is given by $\int_{\pi/3}^{5\pi/3} (1 - 2\cos t)^2 \, dt$.

For a parametric curve, Area $= \int y \, dx = \int y \frac{dx}{dt} \, dt$.

Differentiate x:

$$\frac{dx}{dt} = 1 - 2\cos t$$

Therefore:

$$\text{Area} = \int_{\pi/3}^{5\pi/3} (1 - 2\cos t)(1 - 2\cos t) \, dt = \int_{\pi/3}^{5\pi/3} (1 - 2\cos t)^2 \, dt \quad ✓$$

Part (c): Use this integral to find the exact value of the shaded area.

Expand the integrand:

$$(1 - 2\cos t)^2 = 1 - 4\cos t + 4\cos^2 t$$

Use the double angle formula $\cos^2 t = \frac{1 + \cos 2t}{2}$:

$$= 1 - 4\cos t + 2(1 + \cos 2t) = 3 - 4\cos t + 2\cos 2t$$

Integrate:

$$\int (3 - 4\cos t + 2\cos 2t) \, dt = \Big[3t - 4\sin t + \sin 2t\Big]$$

Apply limits $t = \frac{5\pi}{3}$ and $t = \frac{\pi}{3}$:

At $t = \dfrac{5\pi}{3}$:

$$3 \cdot \frac{5\pi}{3} - 4\sin\frac{5\pi}{3} + \sin\frac{10\pi}{3} = 5\pi - 4\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) = 5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}$$

At $t = \dfrac{\pi}{3}$:

$$3 \cdot \frac{\pi}{3} - 4\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \pi - 4 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}$$

Subtracting:

$$\left(5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) = 4\pi + 4\sqrt{3} - \sqrt{3} = 4\pi + 3\sqrt{3}$$

Area $= 4\pi + 3\sqrt{3}$

Question 2

(Jan 2008, Q7)

Worked Solution

The curve C has parametric equations $x = \ln(t+2)$, $y = \dfrac{1}{t+1}$, $t > -1$.
The region R lies between the curve, the x-axis, $x = \ln 2$ and $x = \ln 4$.

Part (a): Show that the area of R is $\displaystyle\int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{t+2}$$

Change limits: $x = \ln 2 \Rightarrow t+2 = 2 \Rightarrow t = 0$; $\quad x = \ln 4 \Rightarrow t+2 = 4 \Rightarrow t = 2$.

$$\text{Area} = \int_{\ln 2}^{\ln 4} y\,\mathrm{d}x = \int_0^2 \frac{1}{t+1}\cdot\frac{1}{t+2}\,\mathrm{d}t = \int_0^2 \frac{1}{(t+1)(t+2)}\,\mathrm{d}t \quad ✓$$

Part (b): Hence find an exact value for this area.

Use partial fractions:

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$1 = A(t+2) + B(t+1)$. Set $t = -1$: $A = 1$. Set $t = -2$: $B = -1$.

$$\int_0^2 \left(\frac{1}{t+1} - \frac{1}{t+2}\right)\mathrm{d}t = \Big[\ln(t+1) - \ln(t+2)\Big]_0^2$$

$$= \big(\ln 3 - \ln 4\big) - \big(\ln 1 - \ln 2\big) = \ln 3 - \ln 4 + \ln 2 = \ln 3 - \ln 2$$

$$\text{Area} = \ln\left(\frac{3}{2}\right)$$

Part (c): Find a Cartesian equation of C, in the form $y = f(x)$.

From $x = \ln(t+2)$: $e^x = t+2 \Rightarrow t = e^x - 2$.

Substitute into y:

$$y = \frac{1}{t+1} = \frac{1}{e^x - 2 + 1} = \frac{1}{e^x - 1}$$

$$y = \frac{1}{e^x - 1}$$

Part (d): State the domain of values for x.

We need $t > -1$, i.e. $e^x - 2 > -1 \Rightarrow e^x > 1 \Rightarrow x > 0$.

$$x > 0$$

Question 3

(Jun 2008, Q8)

Worked Solution

Curve C: $x = 8\cos t$, $y = 4\sin 2t$, $0 \le t \le \dfrac{\pi}{2}$. Point $P = (4, 2\sqrt{3})$.

Part (a): Find the value of t at P.

$x = 4$: $8\cos t = 4 \Rightarrow \cos t = \frac{1}{2} \Rightarrow t = \dfrac{\pi}{3}$ (in the given range).

Check: $y = 4\sin\left(\dfrac{2\pi}{3}\right) = 4 \cdot \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$. ✓

$t = \dfrac{\pi}{3}$

Part (b): Show that the normal l at P is $y = -x\sqrt{3} + 6\sqrt{3}$.

$$\frac{dx}{dt} = -8\sin t, \quad \frac{dy}{dt} = 8\cos 2t$$
$$\frac{dy}{dx} = \frac{8\cos 2t}{-8\sin t} = \frac{\cos 2t}{-\sin t}$$

At $t = \dfrac{\pi}{3}$: $\dfrac{dy}{dx} = \dfrac{\cos(2\pi/3)}{-\sin(\pi/3)} = \dfrac{-1/2}{-\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$.

Gradient of normal: $m_N = -\sqrt{3}$.

Equation using $P = (4, 2\sqrt{3})$:

$$y - 2\sqrt{3} = -\sqrt{3}(x-4) \implies y = -\sqrt{3}x + 4\sqrt{3} + 2\sqrt{3} = -x\sqrt{3} + 6\sqrt{3} \quad ✓$$

Part (c): Show that the area of R is $\displaystyle\int_{\pi/3}^{\pi/2} 64\sin^2 t\cos t\,dt$.

Region R is bounded by C, the x-axis and $x = 4$.

For t from $\pi/3$ (where $x = 4$) to $\pi/2$ (where $x = 0$):

$$\text{Area} = \int y\,dx = \int_{\pi/3}^{\pi/2} 4\sin 2t \cdot (-8\sin t)\,dt$$

Using $\sin 2t = 2\sin t\cos t$:

$$= \int_{\pi/3}^{\pi/2} 4(2\sin t\cos t)(-8\sin t)\,dt = \int_{\pi/3}^{\pi/2} -64\sin^2 t\cos t\,dt$$

Reversing limits (removing the negative):

$$= \int_{\pi/3}^{\pi/2} 64 \sin^2 t \cos t \, \mathrm{d}t \quad \checkmark$$

Part (d): Find the area of R in the form $a + b\sqrt{3}$.

Use substitution $u = \sin t$, $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \cos t$.

Limits: $t = \pi/3 \Rightarrow u = \sqrt{3}/2$; $\quad t = \pi/2 \Rightarrow u = 1$.

$$A = 64 \int_{\sqrt{3}/2}^{1} u^2 \, \mathrm{d}u = 64 \left[\frac{u^3}{3} \right]_{\sqrt{3}/2}^{1} = 64 \left(\frac{1}{3} - \frac{(\sqrt{3}/2)^3}{3} \right)$$

$$= 64 \left(\frac{1}{3} - \frac{3\sqrt{3}/8}{3} \right) = 64 \left(\frac{1}{3} - \frac{\sqrt{3}}{8} \right) = \frac{64}{3} - 8\sqrt{3}$$

Area $= \dfrac{64}{3} - 8\sqrt{3}$ $\quad$ (so $a = \dfrac{64}{3}$, $b = -8$)

Question 4

(Jan 2010, Q7)

Worked Solution

Curve C: $x = 5t^2 - 4$, $y = t(9 - t^2)$.

Part (a): Find the x-coordinates of A and B.

Set $y = 0$:

$$t(9 - t^2) = t(3 - t)(3 + t) = 0 \implies t = 0,\ 3,\ -3$$

At $t = 0$: $x = 5(0) - 4 = -4$ (point A).
At $t = 3$: $x = 5(9) - 4 = 41$ (point B).
(At $t = -3$: $x = 41$ also, same point B.)

$x_A = -4, \quad x_B = 41$

Part (b): Use integration to find the area of R.

The loop is traced from $t = -3$ to $t = 3$. Since the curve is symmetric about the x-axis, use twice the area above the x-axis (t from 0 to 3):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 10t$$

$$A = 2\int_0^3 y\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = 2\int_0^3 t(9 - t^2)\cdot 10t\,\mathrm{d}t = 20\int_0^3 t^2(9 - t^2)\,\mathrm{d}t$$

$$= 20\int_0^3 (9t^2 - t^4)\,\mathrm{d}t = 20\left[3t^3 - \frac{t^5}{5}\right]_0^3$$

$$= 20\left(3(27) - \frac{243}{5}\right) = 20\left(81 - \frac{243}{5}\right) = 20\cdot\frac{405 - 243}{5} = 20\cdot\frac{162}{5} = \frac{3240}{5} = 648$$

Area of $R = 648$ square units

Question 5 (Jan 2013, Q5)

Worked Solution

Curve C: $x = 1 - \frac{1}{2}t$, $y = 2^t - 1$.

Part (a): Show that A has coordinates $(0, 3)$.

$x = 0$: $1 - \frac{1}{2}t = 0 \Rightarrow t = 2$. Then $y = 2^2 - 1 = 3$. So $A = (0, 3)$. ✓

Part (b): Find the x-coordinate of B.

$y = 0$: $2^t - 1 = 0 \Rightarrow 2^t = 1 \Rightarrow t = 0$.
Then $x = 1 - \frac{1}{2}(0) = 1$.

$x_B = 1$

Part (c): Find an equation of the normal to C at A.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\tfrac{1}{2}, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2^t \ln 2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2^t \ln 2}{-1/2} = -2^{t+1} \ln 2$$

At A where $t = 2$: $m_T = -2^3 \ln 2 = -8 \ln 2$.

Gradient of normal: $m_N = \dfrac{1}{8 \ln 2}$.

Equation using $A = (0, 3)$:

$$y - 3 = \frac{1}{8 \ln 2}(x - 0) \implies y = \frac{x}{8 \ln 2} + 3$$

$y = \dfrac{x}{8 \ln 2} + 3$

Part (d): Use integration to find the exact area of R.

R is bounded by C, $x = -1$ and the x-axis.

Change limits: $x = -1 \Rightarrow -1 = 1 - \frac{1}{2}t \Rightarrow t = 4$; $\quad x = 1 \Rightarrow t = 0$.

$$\text{Area} = \int_{-1}^{1} y \, \mathrm{d}x = \int_{4}^{0} (2^t - 1)\left(-\tfrac{1}{2}\right) \mathrm{d}t = \tfrac{1}{2} \int_{0}^{4} (2^t - 1) \, \mathrm{d}t$$
$$= \tfrac{1}{2} \left[\frac{2^t}{\ln 2} - t \right]_0^4 = \tfrac{1}{2} \left[\left(\frac{16}{\ln 2} - 4 \right) - \left(\frac{1}{\ln 2} - 0 \right) \right]$$
$$= \tfrac{1}{2} \left(\frac{15}{\ln 2} - 4 \right) = \frac{15}{2 \ln 2} - 2$$

$$\text{Area} = \frac{15}{2\ln 2} - 2$$

Question 6

(Edexcel 6666, Sample Paper A3, Q8)

Worked Solution

Ellipse: $x = 5\cos\theta$, $y = 4\sin\theta$, $0 \le \theta < 2\pi$.

Part (a): Show the tangent at $(5\cos\alpha, 4\sin\alpha)$ is $5y\sin\alpha + 4x\cos\alpha = 20$.

$$\frac{dx}{d\theta} = -5\sin\theta, \quad \frac{dy}{d\theta} = 4\cos\theta$$
$$\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta}$$

At $\theta = \alpha$, gradient $= \dfrac{4\cos\alpha}{-5\sin\alpha}$.

Tangent equation:

$$y - 4\sin\alpha = \frac{4\cos\alpha}{-5\sin\alpha}(x - 5\cos\alpha)$$
$$-5\sin\alpha(y - 4\sin\alpha) = 4\cos\alpha(x - 5\cos\alpha)$$
$$-5y\sin\alpha + 20\sin^2\alpha = 4x\cos\alpha - 20\cos^2\alpha$$
$$5y\sin\alpha + 4x\cos\alpha = 20(\sin^2\alpha + \cos^2\alpha) = 20 \quad \checkmark$$

Part (b): Find by integration the area enclosed by the ellipse.

$$\text{Area} = 4\int_0^5 y\,dx$$

Using parametric form, as θ goes from 0 to π (top half), x goes from 5 to -5:

$$\text{Area} = -2\int_0^\pi 4\sin\theta\cdot(-5\sin\theta)\,d\theta = 40\int_0^\pi \sin^2\theta\,d\theta$$

Using $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$:

$$= 40\int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta = 20\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi = 20(\pi - 0) = 20\pi$$

Area $= 20\pi$

Part (c): Show the area between ellipse and parallelogram is $\dfrac{80}{\sin 2\alpha} - 20\pi$.

When $x = 0$: $5\cos\theta = 0$ and $y = \dfrac{4}{\sin\alpha}$ (from tangent equation). So y-intercept of tangent $= \dfrac{4}{\sin\alpha}$.

When $y = 0$: x-intercept $= \dfrac{5}{\cos\alpha}$.

Area of parallelogram $= 4 \times \dfrac{1}{2} \times \dfrac{5}{\cos\alpha} \times \dfrac{4}{\sin\alpha} = \dfrac{40}{\sin\alpha\cos\alpha} = \dfrac{80}{\sin 2\alpha}$.

Area between ellipse and parallelogram:

$$A = \frac{80}{\sin 2\alpha} - 20\pi \quad \checkmark$$

Part (d): Find α where areas of the two types of wood are equal, $0 < \alpha < \pi/4$.

Two equal areas means:

$$\frac{80}{\sin 2\alpha} - 20\pi = 20\pi \implies \frac{80}{\sin 2\alpha} = 40\pi \implies \sin 2\alpha = \frac{2}{\pi}$$

$$2\alpha = \arcsin\left(\frac{2}{\pi}\right) \implies \alpha \approx 0.345$$

$\alpha \approx 0.345$ radians

Question 7

(Edexcel 6666, Sample Paper A6, Q4)

Worked Solution

Cross-section R of a dam. Profile BC: $x = 16t^2 - \pi^2$, $y = 30 \sin 2t$, $\frac{\pi}{4} \le t \le \frac{\pi}{2}$.
Vertices: $A = (0, 0)$, $B = (3\pi^2, 0)$, $C = (0, 30)$.

Part (a): Find an estimate for the area using the straight-line approximation.

Approximating BC as a straight line, R is a triangle:

$$\text{Area} \approx \tfrac{1}{2} \times 3\pi^2 \times 30 = 45\pi^2 \approx 444 \text{ m}^2$$

Estimate $= 45\pi^2 \approx 444$ m^2

Part (b): Find the exact area of R.

$$\frac{dx}{dt} = 32t$$

At $t = \pi/4$: $x = 16(\pi^2/16) - \pi^2 = 0$ (point C projected). At $t = \pi/2$: $x = 16(\pi^2/4) - \pi^2 = 3\pi^2$ (point B).

$$\text{Area} = \int_{\pi/4}^{\pi/2} y \frac{dx}{dt}\, dt = \int_{\pi/4}^{\pi/2} 30 \sin 2t \cdot 32t\, dt = 960 \int_{\pi/4}^{\pi/2} t \sin 2t\, dt$$

Integrate by parts: $u = t$, $\frac{dv}{dt} = \sin 2t$:

$$\int t \sin 2t\, dt = -\frac{t \cos 2t}{2} + \int \frac{\cos 2t}{2}\, dt = -\frac{t \cos 2t}{2} + \frac{\sin 2t}{4}$$

Apply limits:

At $t = \pi/2$: $-\frac{(\pi/2)\cos \pi}{2} + \frac{\sin \pi}{4} = \frac{\pi}{4} + 0 = \frac{\pi}{4}$.

At $t = \pi/4$: $-\frac{(\pi/4)\cos(\pi/2)}{2} + \frac{\sin(\pi/2)}{4} = 0 + \frac{1}{4} = \frac{1}{4}$.

$$\text{Area} = 960 \left(\frac{\pi}{4} - \frac{1}{4} \right) = 960 \cdot \frac{\pi - 1}{4} = 240(\pi - 1)$$

Exact area $= 240(\pi - 1)$ m^2

Part (c): Calculate the percentage error.

$$\text{Percentage error} = \frac{|45\pi^2 - 240(\pi - 1)|}{240(\pi - 1)} \times 100\%$$

$$= \frac{45\pi^2 - 240(\pi - 1)}{240(\pi - 1)} \times 100\% \approx \frac{444.13 - 753.98\ldots}{753.98\ldots} \times 100\%$$

Wait – rechecking: $45\pi^2 \approx 444.1$ and $240(\pi - 1) \approx 240(2.1416) \approx 513.98$.

$$\text{Error} = \frac{|45\pi^2 - 240(\pi - 1)|}{240(\pi - 1)} \times 100 \approx \frac{|444.13 - 513.98|}{513.98} \times 100 \approx 13.6\%$$

Percentage error $\approx 13.6\%$

End of Worked Solutions