Integrating Parametric Functions (From Edexcel 6666)

Q1, (Jan 2006, Q8)

(a)	Solves y = 0 $\Rightarrow$ $\cos t = \frac{1}{2}$ to obtain $t = \frac{\pi}{3}$ *or* $\frac{5\pi}{3}$ (need both for A1)	M1 A1
	Or substitutes **both** values of *t* and shows that $y = 0$	(2)
(b)	$\frac{dx}{dt} = 1 - 2\cos t$	M1 A1
	Area= $\int y\,dx = \int_{\pi/3}^{5\pi/3} (1-2\cos t)(1-2\cos t)dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos t)^2 dt$ * AG	B1 (3)
(c)	Area $= \int 1 - 4\cos t + 4\cos^2 t\,dt$ 3 terms	M1
	$= \int 1 - 4\cos t + 2(\cos 2t + 1)dt$ (use of correct double angle formula)	M1
	$= \int 3 - 4\cos t + 2\cos 2t\,dt$	
	$= [3t - 4\sin t + \sin 2t]$	M1 A1
	Substitutes the two correct limits $t = \frac{5\pi}{3}$ and $\frac{\pi}{3}$ and subtracts.	M1
	$= 4\pi + 3\sqrt{3}$	A1A1 (7)
		[12]

Q2, (Jan 2008, Q7)

(a)	$\left[x=\ln(t+2),\ y=\frac{1}{t+1}\right],\quad \Rightarrow \frac{dx}{dt}=\frac{1}{t+2}$	Must state $\frac{dx}{dt}=\frac{1}{t+2}$	B1
	$\text{Area}(R)=\int_{\ln 2}^{\ln 4}\frac{1}{t+1}\,dx; = \int_0^2\left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right)dt$	Area $=\int\frac{1}{t+1}\,dx$. Ignore limits.	M1;
		$\int\left(\frac{1}{t+1}\right)\times\left(\frac{1}{t+2}\right)dt$. Ignore limits.	A1 **AG**
	Changing limits, when: $x=\ln 2 \Rightarrow \ln 2=\ln(t+2) \Rightarrow 2=t+2 \Rightarrow t=0$ $x=\ln 4 \Rightarrow \ln 4=\ln(t+2) \Rightarrow 4=t+2 \Rightarrow t=2$	changes limits $x\to t$ so that $\ln 2\to 0$ and $\ln 4\to 2$	B1
	Hence, $\text{Area}(R)=\int_0^2\frac{1}{(t+1)(t+2)}\,dt$		**[4]**
b)	$\left(\frac{1}{(t+1)(t+2)}\right)=\frac{A}{(t+1)}+\frac{B}{(t+2)}$	$\frac{A}{(t+1)}+\frac{B}{(t+2)}$ with A and B found	M1
	$1=A(t+2)+B(t+1)$		
	Let $t=-1,\quad 1=A(1) \Rightarrow \underline{A=1}$ Let $t=-2,\quad 1=B(-1) \Rightarrow \underline{B=-1}$	Finds both A and B correctly. Can be implied. (See note below)	A1
	$\int_0^2\frac{1}{(t+1)(t+2)}\,dt=\int_0^2\frac{1}{(t+1)}-\frac{1}{(t+2)}\,dt$		
	$=\left[\ln(t+1)-\ln(t+2)\right]_0^2$	Either $\pm a\ln(t+1)$ or $\pm b\ln(t+2)$ Both ln terms correctly ft.	dM1 A1 $\sqrt{}$
	$=(\ln 3-\ln 4)-(\ln 1-\ln 2)$	Substitutes ***both*** limits of 2 and 0 and subtracts the correct way round.	ddM1
	$=\ln 3-\ln 4+\ln 2=\ln 3-\ln 2=\ln\left(\frac{3}{2}\right)$	$\underline{\ln 3-\ln 4+\ln 2}$ or $\underline{\ln\left(\frac{3}{4}\right)-\ln\left(\frac{1}{2}\right)}$ or $\underline{\ln 3-\ln 2}$ or $\underline{\ln\left(\frac{3}{2}\right)}$ (must deal with ln 1)	A1 aef isw **[6]**

Takes out brackets.

Writing down $\frac{1}{(t+1)(t+2)}=\frac{1}{(t+1)}+\frac{1}{(t+2)}$ means first M1A0 in (b).

Writing down $\frac{1}{(t+1)(t+2)}=\frac{1}{(t+1)}-\frac{1}{(t+2)}$ means first M1A1 in (b).

	$x = \ln(t+2), \quad y = \frac{1}{t+1}$		
(c)	$e^x = t+2 \Rightarrow t = e^x - 2$	Attempt to make $t = ...$ the subject giving $t = e^x - 2$	M1 A1
	$y = \frac{1}{e^x - 2 + 1} \Rightarrow y = \frac{1}{e^x - 1}$	Eliminates t by substituting in y giving $y = \frac{1}{e^x - 1}$	dM1 A1
			[4]
Aliter (c) **Way 2**	$t+1 = \frac{1}{y} \Rightarrow t = \frac{1}{y} - 1$ or $t = \frac{1-y}{y}$ $y(t+1) = 1 \Rightarrow yt + y = 1 \Rightarrow yt = 1 - y \Rightarrow t = \frac{1-y}{y}$	Attempt to make $t = ...$ the subject Giving either $t = \frac{1}{y} - 1$ or $t = \frac{1-y}{y}$	M1 A1
	$x = \ln\left(\frac{1}{y} - 1 + 2\right)$ or $x = \ln\left(\frac{1-y}{y} + 2\right)$	Eliminates t by substituting in x	dM1
	$x = \ln\left(\frac{1}{y} + 1\right)$		
	$e^x = \frac{1}{y} + 1$		
	$e^x - 1 = \frac{1}{y}$		
	$y = \frac{1}{e^x - 1}$	giving $y = \frac{1}{e^x - 1}$	A1
			[4]
(d)	Domain: $\underline{x > 0}$	$\underline{x > 0}$ or just > 0	B1
			[1]
			15 marks

Q3, (Jun 2008, Q8)

(a)	At $P(4, 2\sqrt{3})$ either $\underline{4 = 8\cos t}$ or $\underline{2\sqrt{3} = 4\sin 2t}$	$\underline{4 = 8\cos t}$ or $\underline{2\sqrt{3} = 4\sin 2t}$	M1
	$\Rightarrow$ only solution is $\underline{t = \frac{\pi}{3}}$ where $0 \leqslant t \leqslant \frac{\pi}{2}$	$\underline{t = \frac{\pi}{3}}$ or $\underline{\text{awrt } 1.05}$ (radians) only stated in the range $0 \leqslant t \leqslant \frac{\pi}{2}$	A1
			[2]
(b)	$x = 8\cos t, \quad y = 4\sin 2t$		
	$\frac{dx}{dt} = -8\sin t, \quad \frac{dy}{dt} = 8\cos 2t$	Attempt to differentiate both x and y wrt t to give $\pm p\sin t$ and $\pm q\cos 2t$ respectively	M1
		Correct $\frac{dx}{dt}$ and $\frac{dy}{dt}$	A1
	At P, $\frac{dy}{dx} = \underline{\frac{8\cos\left(\frac{2\pi}{3}\right)}{-8\sin\left(\frac{\pi}{3}\right)}}$	Divides in correct way round and attempts to substitute their value of t (in degrees or radians) into their $\frac{dy}{dx}$ expression.	M1*
	$\left\{= \frac{8\left(-\frac{1}{2}\right)}{(-8)\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}} = \text{awrt } 0.58\right\}$	You may need to check candidate's substitutions for M1* **Note the next two method marks are dependent on M1***	
	Hence $m(\mathbf{N}) = -\sqrt{3}$ or $\frac{-1}{\frac{1}{\sqrt{3}}}$	Uses $m(\mathbf{N}) = -\frac{1}{\text{their } m(\mathbf{T})}$.	dM1*
	N: $y - 2\sqrt{3} = -\sqrt{3}(x - 4)$	Uses $y - 2\sqrt{3} = (\text{their } m_N)(x - 4)$ or finds c using $x = 4$ and $y = 2\sqrt{3}$ and uses $y = (\text{their } m_N)x + "c"$.	dM1*
	N: $\underline{y = -\sqrt{3}x + 6\sqrt{3}}$ **AG**	$\underline{y = -\sqrt{3}x + 6\sqrt{3}}$	A1 cso AG
	or $2\sqrt{3} = -\sqrt{3}(4) + c \Rightarrow c = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$ so N: $\left[\underline{y = -\sqrt{3}x + 6\sqrt{3}}\right]$		
			[6]

. (c)	$A = \int_0^4 y\,dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4\sin 2t.(-8\sin t)\,dt$	attempt at $A = \int \underline{y}\frac{dx}{dt}\,dt$	M1
		correct expression (ignore limits and dt)	A1
	$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32\sin 2t.\sin t\,dt = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -32(2\sin t\cos t).\sin t\,dt$	Seeing $\sin 2t = 2\sin t\cos t$ anywhere in PART (c).	M1
	$A = \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -64.\sin^2 t\cos t\,dt$ $A = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64.\sin^2 t\cos t\,dt$	Correct proof. Appreciation of how the negative sign affects the limits. **Note that the answer is given in the question.**	A1 AG
			[4]
(d)	{Using substitution $u = \sin t \Rightarrow \frac{du}{dt} = \cos t$} {change limits: when $t = \frac{\pi}{3}$, $u = \frac{\sqrt{3}}{2}$ & when $t = \frac{\pi}{2}$, $u = 1$}		
	$A = 64\left[\frac{\sin^3 t}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$ or $A = 64\left[\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{1}$	$k\sin^3 t$ or ku^3 with $u = \sin t$	M1
		Correct integration ignoring limits.	A1
	$A = 64\left[\frac{1}{3} - \left(\frac{1}{3}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}\right)\right]$	Substitutes limits of either $\left(t = \frac{\pi}{2} \text{ and } t = \frac{\pi}{3}\right)$ or $\left(u = 1 \text{ and } u = \frac{\sqrt{3}}{2}\right)$ and subtracts the correct way round.	dM1
	$A = 64\left(\frac{1}{3} - \frac{1}{8}\sqrt{3}\right) = \underline{\frac{64}{3} - 8\sqrt{3}}$	$\underline{\frac{64}{3} - 8\sqrt{3}}$	A1 aef isw
		Aef in the form $a + b\sqrt{3}$, with awrt 21.3 and anything that cancels to $a = \frac{64}{3}$ and $b = -8$.	**[4]**
	(Note that $a = \frac{64}{3}$, $b = -8$)		
			16 marks

Q4, (Jan 2010, Q7)

(a) $y=0 \Rightarrow t(9-t^2)=t(3-t)(3+t)=0$			
$t=0, 3, -3$	Any one correct value	B1	
At $t=0$, $x=5(0)^2-4=-4$	Method for finding one value of x	M1	
At $t=3$, $x=5(3)^2-4=41$			
$\left(\text{At } t=-3,\ x=5(-3)^2-4=41\right)$			
At A, $x=-4$; at B, $x=41$	Both	A1	(3)
(b) $\dfrac{dx}{dt}=10t$	Seen or implied	B1	
$\displaystyle\int y\,dx=\int y\frac{dx}{dt}\,dt=\int t(9-t^2)10t\,dt$		M1 A1	
$\displaystyle=\int(90t^2-10t^4)\,dt$			
$\displaystyle=\frac{90t^3}{3}-\frac{10t^5}{5}\ (+C)\quad\left(=30t^3-2t^5\ (+C)\right)$		A1	
$\displaystyle\left[\frac{90t^3}{3}-\frac{10t^5}{5}\right]_0^3=30\times3^3-2\times3^5\quad(=324)$		M1	
$\displaystyle A=2\int y\,dx=648\quad(\text{units}^2)$		A1	(6)
			[9]

Q5, (Jan 2013, Q5)

	$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$ or $y = e^{t\ln 2} - 1$			
(a)	$\{x = 0 \Rightarrow\}\ 0 = 1 - \frac{1}{2}t \Rightarrow t = 2$	Applies $x = 0$ to obtain a value for t.	M1	
	When $t = 2$, $y = 2^2 - 1 = 3$	Correct value for y.	A1	
				[2]
(b)	$\{y = 0 \Rightarrow\}\ 0 = 2^t - 1 \Rightarrow t = 0$	Applies $y = 0$ to obtain a value for t. (Must be seen in part (b)).	M1	
	When $t = 0$, $x = 1 - \frac{1}{2}(0) = 1$	$x = 1$	A1	
				[2]
(c)	$\frac{dx}{dt} = -\frac{1}{2}$ and either $\frac{dy}{dt} = 2^t \ln 2$ or $\frac{dy}{dt} = e^{t\ln 2}\ln 2$		B1	
	$\frac{dy}{dx} = \frac{2^t \ln 2}{-\frac{1}{2}}$	Attempts their $\frac{dy}{dt}$ divided by their $\frac{dx}{dt}$.	M1	
	At A, $t = "2"$, so $m(\mathbf{T}) = -8\ln 2 \Rightarrow m(\mathbf{N}) = \frac{1}{8\ln 2}$	Applies $t = "2"$ and $m(\mathbf{N}) = \frac{-1}{m(\mathbf{T})}$	M1	
	$y - 3 = \frac{1}{8\ln 2}(x - 0)$ or $y = 3 + \frac{1}{8\ln 2}x$ or equivalent.	See notes.	M1 A1 oe cso	
				[5]
(d)	$\text{Area}(R) = \int (2^t - 1).\left(-\frac{1}{2}\right)dt$	Complete substitution for both y and dx	M1	
	$x = -1 \to t = 4$ and $x = 1 \to t = 0$		B1	
	$= \left\{-\frac{1}{2}\right\}\left(\frac{2^t}{\ln 2} - t\right)$	Either $2^t \to \frac{2^t}{\ln 2}$ or $(2^t - 1) \to \frac{(2^t)}{\pm\alpha(\ln 2)} - t$ or $(2^t - 1) \to \pm\alpha(\ln 2)(2^t) - t$	M1*	
		$(2^t - 1) \to \frac{2^t}{\ln 2} - t$	A1	
	$\left\{-\frac{1}{2}\left[\frac{2^t}{\ln 2} - t\right]_4^0\right\} = -\frac{1}{2}\left(\left(\frac{1}{\ln 2}\right) - \left(\frac{16}{\ln 2} - 4\right)\right)$	**Depends on the previous method mark.** Substitutes their changed limits in t and subtracts either way round.	dM1*	
	$= \frac{15}{2\ln 2} - 2$	$\frac{15}{2\ln 2} - 2$ or equivalent.	A1	
				[6] 15

Q6, (Edexcel 6666, Sample Paper A3, Q8)

(a)	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\cos\theta}{-5\sin\theta}$	M1 A1	
	Equation of tangent is $y - 4\sin\alpha = \frac{4\cos\alpha}{-5\sin\alpha}(x - 5\cos\alpha)$	M1	
	$\therefore 5y\sin\alpha + 4x\cos\alpha = 20(\cos^2\alpha + \sin^2\alpha) = 20 \quad (*)$	A1	(4)
(b)	$\int y\frac{dx}{d\theta}d\theta = -\int 4\sin\theta\ 5\sin\theta\ \ d\theta$	M1	
	$= 10\int(\cos 2\theta - 1)\ \ d\theta$	M1	
	$= [5\sin 2\theta - 10\theta]$	M1	
	Area $= 20\pi$	A1 cso	(4)
(c)	When $x = 0, y = \frac{4}{\sin\alpha}$, or when $y = 0, x = \frac{5}{\cos\alpha}$	B1	
	Area of parallelogram $= 4 \times \frac{10}{\sin\alpha\cos\alpha} = \frac{80}{\sin 2\alpha}$	M1 A1	
	$\therefore A = \frac{80}{\sin 2\alpha} - 20\pi$	A1	(4)
(d)	$\frac{80}{\sin 2\alpha} - 20\pi = 20\pi$	M1 A1	
	$\sin 2\alpha = \frac{2}{\pi}$		
	$\alpha = 0.345$	A1	(3)
		(15 marks)	

Q7, (Edexcel 6666, Sample Paper A6, Q4)

(a)	Area of triangle $= \frac{1}{2}\times 30\times 3\pi^2\ \ (= 444.132)$ Accept 440 or 450	B1	(1)
(b)	Area shaded $= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 30\sin 2t.32t dt$	M1 A1	
	$= \left[-480t\cos 2t + \int 480\cos 2t\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$	M1 A1	
	$= \left[-480t\cos 2t + 240\sin 2t\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$	A1 ft	
	$= 240(\pi - 1)$	M1A1	(7)
(c)	Percentage error $= \frac{240(\pi-1) - estimate}{240(\pi-1)}\times 100 = 13.6\%$ (Accept answers in the range 12.4% to 14.4%)	M1 A1	(2)
		(10 marks)	