Question 1

Worked Solution

$u_{n+1} = 2u_n - 1$, $n \geq 1$, given $u_2 = 9$.

(a) Find u_3 and u_4

$$u_3 = 2u_2 - 1 = 2(9) - 1 = 17$$
$$u_4 = 2u_3 - 1 = 2(17) - 1 = 33$$

$u_3 = 17, \quad u_4 = 33$

(b) Evaluate $\sum_{r=1}^{4} u_r$

First find u_1: from $u_2 = 2u_1 - 1 = 9 \implies u_1 = 5$.

$$\sum_{r=1}^{4} u_r = u_1 + u_2 + u_3 + u_4 = 5 + 9 + 17 + 33$$

$$\sum_{r=1}^{4} u_r = 64$$

Question 2

Worked Solution

$a_1 = 3, \quad a_{n+1} = 2a_n - c, \quad n \geq 1.$

(a) Expression for a_2

$a_2 = 2(3) - c = 6 - c$

(b) Show $a_3 = 12 - 3c$

$$a_3 = 2a_2 - c = 2(6 - c) - c = 12 - 2c - c = 12 - 3c\checkmark$$

(c) Given $\sum_{i=1}^{4} a_i \geq 23$, find range of c

Find a_4:

$$a_4 = 2a_3 - c = 2(12 - 3c) - c = 24 - 6c - c = 24 - 7c$$

Sum:

$$\sum_{i=1}^{4} a_i = 3 + (6 - c) + (12 - 3c) + (24 - 7c) = 45 - 11c$$

Set ≥ 23:

$$45 - 11c \geq 23 \implies -11c \geq -22 \implies c \leq 2$$

$c \leq 2$

Question 3

Worked Solution

$a_1 = k, \quad a_{n+1} = 5a_n + 3, \quad n \geq 1$, where k is a positive integer.

(a) Expression for a_2

$a_2 = 5k + 3$

(b) Show $a_3 = 25k + 18$

$$a_3 = 5a_2 + 3 = 5(5k+3) + 3 = 25k + 15 + 3 = 25k + 18 \checkmark$$

(c)(i) Find $\sum_{r=1}^{4} a_r$ in terms of k

$$a_4 = 5a_3 + 3 = 5(25k+18) + 3 = 125k + 90 + 3 = 125k + 93$$

$$\sum_{r=1}^{4} a_r = k + (5k+3) + (25k+18) + (125k+93) = 156k + 114$$

$$\sum_{r=1}^{4} a_r = 156k + 114$$

(c)(ii) Show divisible by 6

$$156k + 114 = 6(26k+19)$$

Since $6(26k + 19)$ has a factor of 6:

$\sum_{r=1}^{4} a_r = 6(26k + 19)$, which is divisible by 6 for all positive integers k. ✓

Question 4

Worked Solution

$a_1 = 2, \quad a_{n+1} = 3a_n - c.$

(a) Expression for a_2

$a_2 = 3(2) - c = 6 - c$

(b) Given $\sum_{i=1}^{3} a_i = 0$, find c

$$a_3 = 3a_2 - c = 3(6 - c) - c = 18 - 3c - c = 18 - 4c$$

$$\sum_{i=1}^{3} a_i = 2 + (6 - c) + (18 - 4c) = 26 - 5c = 0$$

$c = \dfrac{26}{5} = 5.2$

Question 5

Worked Solution

$a_1 = k, \quad a_{n+1} = 3a_n + 5, \quad n \geq 1.$

(a) Expression for a_2

$a_2 = 3k + 5$

(b) Show $a_3 = 9k + 20$

$$a_3 = 3(3k + 5) + 5 = 9k + 15 + 5 = 9k + 20 \checkmark$$

(c)(i) Find $\sum_{r=1}^{4} a_r$

$$a_4 = 3(9k + 20) + 5 = 27k + 60 + 5 = 27k + 65$$

$$\sum_{r=1}^{4} a_r = k + (3k + 5) + (9k + 20) + (27k + 65) = 40k + 90$$

$\sum_{r=1}^{4} a_r = 40k + 90$

(c)(ii) Show divisible by 10

$$40k + 90 = 10(4k + 9)$$

Since $10(4k + 9)$ has a factor of 10:

$\sum_{r=1}^{4} a_r = 10(4k + 9)$, divisible by 10 for all positive integers k. ✓

Question 6

Worked Solution

$a_1 = k, \quad a_{n+1} = 2a_n - 7, \quad n \geq 1.$

(a) Expression for a_2

$a_2 = 2k - 7$

(b) Show $a_3 = 4k - 21$

$$a_3 = 2(2k - 7) - 7 = 4k - 14 - 7 = 4k - 21 \checkmark$$

(c) Given $\sum_{r=1}^{4} a_r = 43$**, find** k

$$a_4 = 2(4k - 21) - 7 = 8k - 42 - 7 = 8k - 49$$

$$\sum_{r=1}^{4} a_r = k + (2k - 7) + (4k - 21) + (8k - 49) = 15k - 77 = 43$$

$$15k = 120$$

$k = 8$

Question 7

Worked Solution

$a_{n+1} = \sqrt{a_n^2 + 3}$, $n \geq 1$, $\quad a_1 = 2$.

(a) Find a_2 and a_3 in surd form

$$a_2 = \sqrt{a_1^2 + 3} = \sqrt{4+3} = \sqrt{7}$$

$$a_3 = \sqrt{a_2^2 + 3} = \sqrt{7+3} = \sqrt{10}$$

$a_2 = \sqrt{7}, \quad a_3 = \sqrt{10}$

(b) Show $a_5 = 4$

$$a_4 = \sqrt{a_3^2 + 3} = \sqrt{10+3} = \sqrt{13}$$

$$a_5 = \sqrt{a_4^2 + 3} = \sqrt{13+3} = \sqrt{16} = 4\checkmark$$

Question 8

Worked Solution

$x_1 = 1, \quad x_{n+1} = x_n(p + x_n).$

(a) Find x_2 in terms of p

$$x_2 = x_1(p + x_1) = 1(p + 1) = p + 1$$

$x_2 = p + 1$

(b) Show $x_3 = 1 + 3p + 2p^2$

$$x_3 = x_2(p + x_2) = (p + 1)(p + p + 1) = (p + 1)(2p + 1)$$
$$= 2p^2 + p + 2p + 1 = 2p^2 + 3p + 1 = 1 + 3p + 2p^2 \checkmark$$

(c) Given $x_3 = 1$, find p

$$1 + 3p + 2p^2 = 1 \implies 3p + 2p^2 = 0 \implies p(3 + 2p) = 0$$

Since $p \neq 0$:

$p = -\dfrac{3}{2}$

(d) Write down x_{2008}

With $p = -\frac{3}{2}$:

$$x_1 = 1, \quad x_2 = -\tfrac{1}{2}, \quad x_3 = 1, \quad x_4 = -\tfrac{1}{2}, \quad \ldots$$

The sequence alternates: odd terms $= 1$, even terms $= -\frac{1}{2}$. Since 2008 is even:

$x_{2008} = -\dfrac{1}{2}$

Question 9

Worked Solution

$a_{n+1} = \dfrac{k(a_n + 2)}{a_n}$, $\quad a_1 = 2$. Sequence is periodic of order 3.

(a) Show $k^2 + k - 2 = 0$

Generate terms:

$$a_2 = \frac{k(2+2)}{2} = 2k$$
$$a_3 = \frac{k(2k+2)}{2k} = \frac{k \cdot 2(k+1)}{2k} = k + 1$$
$$a_4 = \frac{k(k+1+2)}{k+1} = \frac{k(k+3)}{k+1}$$

For period 3: $a_4 = a_1 = 2$:

$$\frac{k(k+3)}{k+1} = 2 \implies k^2 + 3k = 2k + 2 \implies k^2 + k - 2 = 0 \checkmark$$

(b) Explain why $k \neq 1$

$$k^2 + k - 2 = 0 \implies (k+2)(k-1) = 0 \implies k = 1 \text{ or } k = -2$$

If $k = 1$: $a_1 = 2$, $a_2 = 2$, $a_3 = 2$ – all terms equal, so the sequence has period 1, not period 3.

$k \neq 1$ because when $k = 1$ all terms are equal (period 1, not 3).

(c) Find $\sum_{r=1}^{80} a_r$ **with** $k = -2$

With $k = -2$: $a_1 = 2$, $a_2 = 2(-2) = -4$, $a_3 = -2 + 1 = -1$. Period is 3 with sum per cycle $= 2 + (-4) + (-1) = -3$.

$80 = 3 \times 26 + 2$, so:

$$\sum_{r=1}^{80} a_r = 26 \times (-3) + a_1 + a_2 = -78 + 2 + (-4)$$

$$\sum_{r=1}^{80} a_r = -80$$

Question 10

Worked Solution

$a_1 = 1, \quad a_{n+1} = \dfrac{k(a_n + 1)}{a_n}, \quad n \geq 1.$

(a) Expressions for a_2 and a_3

$$a_2 = \frac{k(1+1)}{1} = 2k$$

$$a_3 = \frac{k(2k+1)}{2k} = \frac{k(2k+1)}{2k} = \frac{2k+1}{2}$$

$a_2 = 2k, \quad a_3 = \dfrac{2k+1}{2}$

(b) Given $\sum_{r=1}^{3} a_r = 10$, find exact k

$$1 + 2k + \frac{2k+1}{2} = 10$$

Multiply through by 2:

$$2 + 4k + 2k + 1 = 20 \implies 6k + 3 = 20 \implies 6k = 17$$

$k = \dfrac{17}{6}$

Question 11

Worked Solution

$a_1 = 1, \quad a_{n+1} = \dfrac{k(a_n + 1)}{a_n}, \quad n \geq 1$ (same recurrence as Q10).

(a) Expressions for a_2 and a_3

$a_2 = 2k, \quad a_3 = \dfrac{2k+1}{2}$

(b) Given $\sum_{r=1}^{3} a_r = 10$, find exact k

$$1 + 2k + \frac{2k+1}{2} = 10 \implies 6k = 17$$

$k = \dfrac{17}{6}$

Question 12

Worked Solution

$x_1 = 1, \quad x_{n+1} = ax_n - 3, \quad n \geq 1.$

(a) Expression for x_2

$x_2 = a(1) - 3 = a - 3$

(b) Show $x_3 = a^2 - 3a - 3$

$$x_3 = a\,x_2 - 3 = a(a-3) - 3 = a^2 - 3a - 3 \checkmark$$

(c) Given $x_3 = 7$, find possible values of a

$$a^2 - 3a - 3 = 7 \implies a^2 - 3a - 10 = 0$$

$$(a-5)(a+2) = 0$$

$a = 5 \quad \text{or} \quad a = -2$

End of Worked Solutions