Question 1 (Jun 2007, Q6)

Worked Solution

Curve: $x^2 + 3xy + 4y^2 = 58$. Find normal at $(2,3)$ in the form $ax + by + c = 0$.

Differentiate implicitly:

$$2x + 3y + 3x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$(3x + 8y)\frac{dy}{dx} = -2x - 3y$$

$$\frac{dy}{dx} = \frac{-2x - 3y}{3x + 8y}$$

At $(2,3)$:

$$\frac{dy}{dx} = \frac{-4 - 9}{6 + 24} = \frac{-13}{30}$$

Gradient of normal $= \frac{30}{13}$.

Normal through $(2,3)$:

$$y - 3 = \frac{30}{13}(x - 2) \implies 13y - 39 = 30x - 60 \implies 30x - 13y - 21 = 0$$

$30x - 13y - 21 = 0$

Question 2 (Jan 2008, Q4)

Worked Solution

Curve: $x^3 + 4x^2y + y^3 = 6$. Find normal at $(1, 1)$ in form $ax + by + c = 0$.

Differentiate implicitly:

$$3x^2 + 8xy + 4x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(4x^2 + 3y^2)\frac{dy}{dx} = -3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{-3x^2 - 8xy}{4x^2 + 3y^2}$$

At $(1, 1)$:

$$\frac{dy}{dx} = \frac{-3 - 8}{4 + 3} = -\frac{11}{7}$$

Gradient of normal $= \dfrac{7}{11}$.

Normal through $(1, 1)$:

$$y - 1 = \frac{7}{11}(x - 1) \implies 11y - 11 = 7x - 7 \implies 7x - 11y + 4 = 0$$

$7x - 11y + 4 = 0$

Question 3 (Jan 2009, Q8)

Worked Solution

Curve: $x^3 + y^3 = 6xy$.

(i) Find $\frac{dy}{dx}$

Differentiate implicitly, treating $6xy$ with the product rule:

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

(ii) Show $\left(2^{4/3}, 2^{5/3}\right)$ lies on the curve and $\frac{dy}{dx} = 0$ there

Let $x = 2^{4/3}$, $y = 2^{5/3}$:

$$x^3 = 2^4 = 16, \quad y^3 = 2^5 = 32$$

$$x^3 + y^3 = 48, \quad 6xy = 6 \times 2^{4/3} \times 2^{5/3} = 6 \times 2^{9/3} = 6 \times 8 = 48 \checkmark$$

Numerator of $\frac{dy}{dx}$: $6y - 3x^2 = 6 \times 2^{5/3} - 3 \times 2^{8/3} = 6 \times 2^{5/3} - 6 \times 2^{5/3} = 0$ ✓

(iii) Point (a, a) with $a > 0$: find a and gradient

Substitute $y = x$ into curve: $x^3 + x^3 = 6x^2 \implies 2x^3 = 6x^2 \implies x = 3$ (since $x > 0$). So $a = 3$, point is $(3, 3)$.

$$\frac{dy}{dx} = \frac{2(3) - 9}{9 - 6} = \frac{-3}{3}$$

$a = 3$. Gradient at $(3, 3)$ is -1.

Question 4 (Jun 2009, Q8)

Worked Solution

(i) Show $\frac{dy}{dx} = \frac{28x - 7y}{7x - 2y}$ **for** $14x^2 - 7xy + y^2 = 2$

Differentiate implicitly:

$$28x - 7y - 7x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(2y - 7x)\frac{dy}{dx} = 7y - 28x$$

$$\frac{dy}{dx} = \frac{7y - 28x}{2y - 7x} = \frac{-(28x - 7y)}{-(7x - 2y)} = \frac{28x - 7y}{7x - 2y} \checkmark$$

(ii) Find coordinates of N, where tangents at L and M meet

At $x = 1$: $14 - 7y + y^2 = 2 \implies y^2 - 7y + 12 = 0 \implies (y - 3)(y - 4) = 0$, so $y = 3$ or $y = 4$.

At $L(1, 3)$: $\frac{dy}{dx} = \frac{28 - 21}{7 - 6} = 7$. Tangent: $y - 3 = 7(x - 1) \implies y = 7x - 4$.

At $M(1, 4)$: $\frac{dy}{dx} = \frac{28 - 28}{7 - 8} = 0$. Tangent: $y = 4$.

Intersection: $7x - 4 = 4 \implies x = \frac{8}{7}$.

$N = \left(\frac{8}{7}, 4\right)$

Question 5 (Jun 2010, Q5)

Worked Solution

Curve: $x^2 + 4xy + 2y^2 + 18 = 0$. Find the two stationary points.

Differentiate implicitly:

$$2x + 4y + 4x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$(4x + 4y)\frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x - 4y}{4x + 4y} = \frac{-(x + 2y)}{2(x + y)}$$

At stationary points, $\frac{dy}{dx} = 0 \implies x + 2y = 0 \implies x = -2y$.

Substitute into curve:

$$(-2y)^2 + 4(-2y)y + 2y^2 + 18 = 0$$

$$4y^2 - 8y^2 + 2y^2 + 18 = 0 \implies -2y^2 = -18 \implies y^2 = 9$$

$$y = \pm 3, \quad x = \mp 6$$

Stationary points: $(6, -3)$ and $(-6, 3)$

Question 6 (Jan 2013, Q3)

Worked Solution

Curve: $xy^2 = x^2 + 1$. Find $\frac{dy}{dx}$ and stationary points.

Differentiate implicitly using the product rule on xy^2:

$$y^2 + 2xy\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x - y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{2x - y^2}{2xy}$$

At stationary points, $\frac{dy}{dx} = 0 \implies 2x - y^2 = 0 \implies y^2 = 2x$.

Substitute into curve: $x(2x) = x^2 + 1 \implies x^2 = 1 \implies x = \pm 1$.

Since $y^2 = 2x \geq 0$, we need $x = 1$, giving $y^2 = 2$, so $y = \pm\sqrt{2}$.

Stationary points: $\left(1, \sqrt{2}\right)$ and $\left(1, -\sqrt{2}\right)$

Question 7 (Jun 2015, Q7)

Worked Solution

Curve: $(x+y)^2 = xy^2$. Find gradient where $x = 1$.

Differentiate implicitly using chain rule on LHS and product rule on RHS:

$$2(x+y)\left(1+\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$$

First find y when $x = 1$: $(1+y)^2 = y^2 \implies 1 + 2y + y^2 = y^2 \implies 2y = -1 \implies y = -\frac{1}{2}$.

At $\left(1, -\frac{1}{2}\right)$, $x + y = \frac{1}{2}$:

$$2\left(\tfrac{1}{2}\right)\left(1+\frac{dy}{dx}\right) = \left(-\tfrac{1}{2}\right)^2 + 2(1)\left(-\tfrac{1}{2}\right)\frac{dy}{dx}$$

$$1 + \frac{dy}{dx} = \frac{1}{4} - \frac{dy}{dx}$$

$$2\frac{dy}{dx} = \frac{1}{4} - 1 = -\frac{3}{4}$$

$$\frac{dy}{dx} = -\frac{3}{8}$$

Question 8 (Jun 2016, Q3)

Worked Solution

Given $y \sin 2x + \frac{1}{x} + y^2 = 5$, find $\frac{dy}{dx}$.

Differentiate implicitly:

$$\frac{dy}{dx}\sin 2x + 2y\cos 2x - \frac{1}{x^2} + 2y\frac{dy}{dx} = 0$$

$$(\sin 2x + 2y)\frac{dy}{dx} = \frac{1}{x^2} - 2y\cos 2x$$

$$\frac{dy}{dx} = \frac{\frac{1}{x^2} - 2y\cos 2x}{\sin 2x + 2y} = \frac{1 - 2x^2 y\cos 2x}{x^2(\sin 2x + 2y)}$$

Question 9 (Jun 2014, Q6)

Worked Solution

Curve: $x^2 + y^3 - 8x - 12y = 4$. At P and Q the tangent is parallel to the y-axis.

Differentiate implicitly:

$$2x + 3y^2\frac{dy}{dx} - 8 - 12\frac{dy}{dx} = 0$$

$$(3y^2 - 12)\frac{dy}{dx} = 8 - 2x$$

$$\frac{dy}{dx} = \frac{8 - 2x}{3y^2 - 12}$$

Tangent parallel to y-axis means $\frac{dx}{dy} = 0$, i.e. denominator of $\frac{dy}{dx} \to \infty$, which means $\frac{dx}{dy} = 0$:

$$3y^2 - 12 = 0 \implies y^2 = 4 \implies y = \pm 2$$

Substitute $y = 2$ into curve:

$$x^2 + 8 - 8x - 24 = 4 \implies x^2 - 8x - 20 = 0 \implies (x - 10)(x + 2) = 0$$

So $x = 10$ or $x = -2$.

Substitute $y = -2$:

$$x^2 - 8 - 8x + 24 = 4 \implies x^2 - 8x + 12 = 0 \implies (x - 6)(x - 2) = 0$$

But we need points where the tangent is parallel to the y-axis; checking: from the diagram P and Q are the leftmost and rightmost points.

With $y = 2$: $x = -2$ or $x = 10$. With $y = -2$: $x = 2$ or $x = 6$.

The points where the tangent is parallel to the y-axis (i.e. $\frac{dx}{dy} = 0$):

$\frac{dx}{dy} = 0 \implies 8 - 2x = 0 \implies x = 4$.

Substitute $x = 4$ into curve:

$$16 + y^3 - 32 - 12y = 4 \implies y^3 - 12y - 20 = 0$$

Try $y = -2$: $-8 + 24 - 20 = -4 \neq 0$. Try $y = 2$: $8 - 24 - 20 \neq 0$.

Re-examining: parallel to y-axis means $\frac{dy}{dx}$ is undefined (denominator $= 0$ and numerator $\neq 0$), so $3y^2 - 12 = 0$, $y = \pm 2$.

At $y = 2$, $x = 10$ or $x = -2$. Check $8 - 2x \neq 0$: at $x = 10$, $8 - 20 \neq 0$ ✓; at $x = -2$, $8 + 4 \neq 0$ ✓.

$P = (-2, 2)$ and $Q = (10, 2)$
(Also valid: the two points with $y = 2$ since $y = -2$ gives $x = 2$ or 6 but these are not the leftmost/rightmost shown in the diagram.)

End of Worked Solutions