Question 1

Worked Solution

Curve: $\ln y = 2x \ln x, \quad x > 0,\, y > 0$. Find gradient at $x = 2$.

Differentiate implicitly with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x + 2x \cdot \frac{1}{x} = 2\ln x + 2$$

Find y when $x = 2$: $\ln y = 2(2)\ln 2 = 4\ln 2$, so $y = e^{4\ln 2} = 16$.

At the point $(2,\, 16)$:

$$\frac{1}{16}\frac{dy}{dx} = 2\ln 2 + 2$$

$$\frac{dy}{dx} = 16(2\ln 2 + 2) = 32\ln 2 + 32$$

$$\frac{dy}{dx} = 16(2 + 2\ln 2)$$

Question 2

Worked Solution

Curve: $x^3 - 4y^2 = 12xy$.

(a) Coordinates where $x = -8$

Substitute $x = -8$:

$$(-8)^3 - 4y^2 = 12(-8)y \implies -512 - 4y^2 = -96y$$

$$4y^2 - 96y + 512 = 0 \implies y^2 - 24y + 128 = 0$$

$$(y - 16)(y - 8) = 0 \implies y = 16 \text{ or } y = 8$$

$(-8, 8)$ and $(-8, 16)$

(b) Gradient at each point

Differentiate $x^3 - 4y^2 = 12xy$ implicitly:

$$3x^2 - 8y\frac{dy}{dx} = 12y + 12x\frac{dy}{dx}$$

$$3x^2 - 12y = (12x + 8y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$$

At $(-8, 8)$:

$$\frac{dy}{dx} = \frac{3(64) - 12(8)}{12(-8) + 8(8)} = \frac{192 - 96}{-96 + 64} = \frac{96}{-32}$$

At $(-8, 16)$:

$$\frac{dy}{dx} = \frac{3(64) - 12(16)}{12(-8) + 8(16)} = \frac{192 - 192}{-96 + 128} = \frac{0}{32}$$

At $(-8, 8)$: gradient $= -3$. At $(-8, 16)$: gradient $= 0$.

Question 3

Worked Solution

Curve: $3x^2 - y^2 + xy = 4$. Gradient of tangent is $\frac{8}{3}$ at P and Q.

(a) Show $y - 2x = 0$ at P and Q

Differentiate implicitly:

$$6x - 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$(x - 2y)\frac{dy}{dx} = -6x - y$$

$$\frac{dy}{dx} = \frac{-6x - y}{x - 2y}$$

Set $\dfrac{dy}{dx} = \dfrac{8}{3}$:

$$\frac{-6x - y}{x - 2y} = \frac{8}{3}$$

$$3(-6x - y) = 8(x - 2y)$$

$$-18x - 3y = 8x - 16y$$

$$13y = 26x \implies y = 2x$$

$y - 2x = 0$ at P and Q ✓

(b) Find coordinates of P and Q

Substitute $y = 2x$ into the curve equation:

$$3x^2 - (2x)^2 + x(2x) = 4 \implies 3x^2 - 4x^2 + 2x^2 = 4 \implies x^2 = 4$$

$$x = \pm 2, \quad y = \pm 4$$

$P = (2, 4)$ and $Q = (-2, -4)$

Question 4

Worked Solution

Curve C: $\cos 2x + \cos 3y = 1, \quad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}, \quad 0 \leq y \leq \frac{\pi}{6}.$

(a) Find $\frac{dy}{dx}$

Differentiate implicitly:

$$-2\sin 2x - 3\sin 3y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2\sin 2x}{3\sin 3y}$$

(b) Find y at P where $x = \frac{\pi}{6}$

$$\cos\left(\frac{2\pi}{6}\right) + \cos 3y = 1 \implies \cos\left(\frac{\pi}{3}\right) + \cos 3y = 1 \implies \frac{1}{2} + \cos 3y = 1$$

$$\cos 3y = \frac{1}{2} \implies 3y = \frac{\pi}{3} \implies y = \frac{\pi}{9}$$

$$y = \frac{\pi}{9}$$

(c) Equation of tangent at $P = \left(\frac{\pi}{6}, \frac{\pi}{9}\right)$

$$\frac{dy}{dx} = -\frac{2\sin\left(\frac{\pi}{3}\right)}{3\sin\left(\frac{\pi}{3}\right)} = -\frac{2}{3}$$

Tangent: $y - \frac{\pi}{9} = -\frac{2}{3}\left(x - \frac{\pi}{6}\right)$

Multiply through by 9:

$$9y - \pi = -6\left(x - \frac{\pi}{6}\right) = -6x + \pi$$

$$6x + 9y - 2\pi = 0$$

$$6x + 9y - 2\pi = 0$$

Question 5

Worked Solution

Curve C: $x^2 - 3xy - 4y^2 + 64 = 0$.

(a) Find $\frac{dy}{dx}$

Differentiate implicitly:

$$2x - \left(3y + 3x\frac{dy}{dx}\right) - 8y\frac{dy}{dx} = 0$$

$$2x - 3y + (-3x - 8y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2x - 3y}{3x + 8y}$$

$$\frac{dy}{dx} = \frac{2x - 3y}{3x + 8y}$$

(b) Find coordinates where $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0 \implies 2x - 3y = 0 \implies y = \frac{2x}{3}$

Substitute into curve:

$$x^2 - 3x\left(\frac{2x}{3}\right) - 4\left(\frac{2x}{3}\right)^2 + 64 = 0$$

$$x^2 - 2x^2 - \frac{16x^2}{9} + 64 = 0$$

$$-x^2 - \frac{16x^2}{9} + 64 = 0 \implies -\frac{25x^2}{9} = -64 \implies x^2 = \frac{576}{25}$$

$$x = \pm\frac{24}{5}, \quad y = \pm\frac{16}{5}$$

$$\left(\frac{24}{5}, \frac{16}{5}\right) \text{ and } \left(-\frac{24}{5}, -\frac{16}{5}\right)$$

Question 6

Worked Solution

$x^2 + y^2 + 10x + 2y - 4xy = 10$

(a) Find $\frac{dy}{dx}$

Differentiate implicitly:

$$2x + 2y\frac{dy}{dx} + 10 + 2\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} = 0$$

$$(2y + 2 - 4x)\frac{dy}{dx} = -2x - 10 + 4y$$

$$\frac{dy}{dx} = \frac{-2x - 10 + 4y}{2y + 2 - 4x} = \frac{4y - 2x - 10}{2y - 4x + 2}$$

Simplify by dividing numerator and denominator by 2:

$$\frac{dy}{dx} = \frac{2y - x - 5}{y - 2x + 1}$$

(b) Find values of y for which $\frac{dy}{dx} = 0$

$2y - x - 5 = 0 \implies x = 2y - 5$

Substitute into original equation:

$$(2y - 5)^2 + y^2 + 10(2y - 5) + 2y - 4(2y - 5)y = 10$$

$$4y^2 - 20y + 25 + y^2 + 20y - 50 + 2y - 8y^2 + 20y = 10$$

$$-3y^2 + 22y - 25 = 10$$

$$3y^2 - 22y + 35 = 0$$

$$(3y - 7)(y - 5) = 0$$

$$y = \frac{7}{3} \quad \text{or} \quad y = 5$$

Question 7

Worked Solution

Curve: $x^2 - 2xy + 3y^2 = 50$.

(a) Show $\dfrac{dy}{dx} = \dfrac{y-x}{3y-x}$

Differentiate implicitly:

$$2x - 2\left(y + x\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$$

$$2x - 2y + (-2x + 6y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2y-2x}{6y-2x} = \frac{y-x}{3y-x}\checkmark$$

(b) Find exact coordinates of P (furthest west)

At P the tangent is parallel to the y-axis, so $\dfrac{dx}{dy} = 0$, i.e. $3y - x = 0 \implies x = 3y$.

Substitute into curve:

$$(3y)^2 - 2(3y)y + 3y^2 = 50 \implies 9y^2 - 6y^2 + 3y^2 = 50 \implies 6y^2 = 50$$

$$y^2 = \frac{25}{3} \implies y = \pm\frac{5}{\sqrt{3}} = \pm\frac{5\sqrt{3}}{3}$$

P is furthest west (most negative x-value), so $x = 3y < 0 \implies y < 0$:

$$y = -\frac{5\sqrt{3}}{3}, \quad x = 3 \times \left(-\frac{5\sqrt{3}}{3}\right) = -5\sqrt{3}$$

$$P = \left(-5\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)$$

(c) How to find the point furthest north

Set $\dfrac{dy}{dx} = 0$, i.e. $y - x = 0 \implies y = x$. Substitute $y = x$ into the curve equation and solve for x, then take the positive y value (largest y).

Question 8

Worked Solution

Curve C: $px^3 + qxy + 3y^2 = 26$.

(a) Find $\frac{dy}{dx}$ in the form $\frac{apx^2 + bqy}{qx + cy}$

Differentiate implicitly:

$$3px^2 + qy + qx\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$(qx + 6y)\frac{dy}{dx} = -3px^2 - qy$$

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}, \quad \text{so } a = -3,\ b = -1,\ c = 6$$

(b) Find p and q

$P(-1, -4)$ lies on C:

$$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26 \implies -p + 4q + 48 = 26 \implies -p + 4q = -22 \quad (1)$$

Normal at P has equation $19x + 26y + 123 = 0$, so gradient of normal $= -\frac{19}{26}$.

Gradient of tangent at $P = \frac{26}{19}$.

Using $\frac{dy}{dx}$ at $(-1, -4)$:

$$\frac{-3p(1) - q(-4)}{q(-1) + 6(-4)} = \frac{26}{19}$$

$$\frac{-3p + 4q}{-q - 24} = \frac{26}{19}$$

$$19(-3p + 4q) = 26(-q - 24)$$

$$-57p + 76q = -26q - 624$$

$$-57p + 102q = -624 \quad (2)$$

From (1): $p = 4q + 22$. Substitute into (2):

$$-57(4q+22)+102q = -624 \implies -228q-1254+102q = -624 \implies -126q = 630 \implies q = -5$$

$$p = 4(-5) + 22 = 2$$

$$p = 2, \quad q = -5$$

Question 9

Worked Solution

Curve C: $3^{x-1} + xy - y^2 + 5 = 0$.

Show that $\frac{dy}{dx}$ at $(1, 3)$ can be written as $\frac{1}{\lambda}\ln(\mu e^3)$ where λ, μ are integers.

Differentiate implicitly (note $\frac{d}{dx}(3^{x-1}) = 3^{x-1}\ln 3$):

$$3^{x-1}\ln 3 + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$(x - 2y)\frac{dy}{dx} = -3^{x-1}\ln 3 - y$$

At $(1, 3)$:

$$(1-6)\frac{dy}{dx} = -3^0\ln 3 - 3 = -\ln 3 - 3$$

$$-5\,\frac{dy}{dx} = -\ln 3 - 3$$

$$\frac{dy}{dx} = \frac{\ln 3 + 3}{5}$$

Now use $3 = \ln e^3$:

$$\frac{dy}{dx} = \frac{\ln 3 + \ln e^3}{5} = \frac{\ln(3e^3)}{5} = \frac{1}{5}\ln(3e^3)$$

$$\frac{dy}{dx} = \frac{1}{5}\ln(3e^3), \quad \text{so } \lambda = 5,\ \mu = 3$$

Question 10

Worked Solution

Curve C: $4x^2 - y^3 - 4xy + 2^y = 0$. Point $P(-2, 4)$ lies on C.

(a) Find exact value of $\frac{dy}{dx}$ at P

Differentiate implicitly ($\frac{d}{dx}(2^y) = 2^y \ln 2 \cdot \frac{dy}{dx}$):

$$8x - 3y^2\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} + 2^y \ln 2 \cdot \frac{dy}{dx} = 0$$

At $(-2, 4)$, note $2^4 = 16$:

$$8(-2) - 3(16)\frac{dy}{dx} - 4(4) - 4(-2)\frac{dy}{dx} + 16 \ln 2 \cdot \frac{dy}{dx} = 0$$

$$-16 - 48\frac{dy}{dx} - 16 + 8\frac{dy}{dx} + 16 \ln 2 \cdot \frac{dy}{dx} = 0$$

$$-32 + (-40 + 16 \ln 2)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{32}{-40 + 16 \ln 2} = \frac{32}{16(\ln 2 - \frac{5}{2})} = \frac{2}{\ln 2 - \frac{5}{2}}$$

Multiply numerator and denominator by 2:

$$\frac{dy}{dx} = \frac{4}{2 \ln 2 - 5} \quad \text{(equivalently } \frac{32}{16 \ln 2 - 40}\text{)}$$

(b) Find y-coordinate of A, where normal meets y-axis

Gradient of tangent at $P = \frac{4}{2 \ln 2 - 5}$.

Gradient of normal $m_N = -\frac{2 \ln 2 - 5}{4} = \frac{5 - 2 \ln 2}{4}$.

Normal line through $P(-2, 4)$:

$$y - 4 = \frac{5 - 2 \ln 2}{4}(x - (-2)) = \frac{5 - 2 \ln 2}{4}(x + 2)$$

At y-axis, $x = 0$:

$$y - 4 = \frac{5 - 2 \ln 2}{4}(2) = \frac{5 - 2 \ln 2}{2}$$

$$y = 4 + \frac{5 - 2 \ln 2}{2} = \frac{8 + 5 - 2 \ln 2}{2} = \frac{13 - 2 \ln 2}{2}$$

$$y = \frac{13}{2} - \ln 2, \quad \text{i.e. } p = \frac{13}{2},\ q = -1$$

Question 11

Worked Solution

Curve C: $2^x + y^2 = 2xy$. Find exact $\dfrac{dy}{dx}$ at $(3, 2)$.

Differentiate implicitly (note $\dfrac{d}{dx}(2^x) = \ln 2 \cdot 2^x$):

$$\ln 2 \cdot 2^x + 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$(2y - 2x)\frac{dy}{dx} = 2y - \ln 2 \cdot 2^x$$

At $(3, 2)$, note $2^3 = 8$:

$$(4 - 6)\frac{dy}{dx} = 4 - 8\ln 2$$

$$-2\,\frac{dy}{dx} = 4 - 8\ln 2$$

$$\frac{dy}{dx} = \frac{4 - 8\ln 2}{-2} = 4\ln 2 - 2$$

$$\frac{dy}{dx} = 4\ln 2 - 2$$

End of Worked Solutions