Question 1

Worked Solution

(a) Prove that $S_n = \dfrac{a(1-r^n)}{1-r}$

Write out the sum and the sum multiplied by r:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtract: $S_n - rS_n = a - ar^n$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

(b) Find the common ratio

Third term: $ar^2 = 5.4$, Fifth term: $ar^4 = 1.944$

Divide fifth by third:

$$r^2 = \frac{1.944}{5.4} = 0.36 \implies r = \pm 0.6$$

Since all terms are positive:

$r = 0.6$

(c) Find the first term

Using $ar^2 = 5.4$:

$$a(0.6)^2 = 5.4 \implies a \times 0.36 = 5.4 \implies a = 15$$

$a = 15$

(d) Find the sum to infinity

$$S_\infty = \frac{a}{1-r} = \frac{15}{1-0.6} = \frac{15}{0.4}$$

$S_\infty = 37.5$

Question 2

Worked Solution

(a) Show the predicted profit in 2016 is £138 915

2013 is Year 1, so 2016 is Year 4. The profit in Year 4 is:

$$120000 \times (1.05)^3 = 120000 \times 1.157625 = 138915✓$$

(b) Find the first year in which yearly profit exceeds £200 000

$$120000 \times (1.05)^{n-1} > 200000$$

$$(1.05)^{n-1} > \frac{200000}{120000} = \frac{5}{3}$$

Taking logarithms:

$$(n-1)\log(1.05) > \log\left(\tfrac{5}{3}\right)$$

$$n-1 > \frac{\log(5/3)}{\log(1.05)} = \frac{0.22185\ldots}{0.02119\ldots} = 10.472\ldots$$

$$n > 11.47\ldots \implies n = 12$$

Year 12 corresponds to $2013 + 11 =$

2024 is the first year in which the predicted profit exceeds £200 000.

(c) Total predicted profit 2013–2023 inclusive (11 years)

$$S_{11} = \frac{120000(1-1.05^{11})}{1-1.05} = \frac{120000(1-1.05^{11})}{-0.05}$$

$$1.05^{11} = 1.71033936\ldots$$

$$S_{11} = \frac{120000(1-1.71034)}{-0.05} = \frac{120000 \times (-0.71034)}{-0.05} = \frac{-85240.8}{-0.05}$$

$S_{11} = £1\,704\,814$ (to the nearest pound)

Question 3

Worked Solution

(a) Find the common ratio

$$r = \frac{12}{18}$$

$r = \frac{2}{3}$

(b) Find the value of p

$$p = 12 \times \frac{2}{3}$$

$p = 8$

(c) Sum of first 15 terms

$$S_{15} = \frac{18\left(1 - \left(\frac{2}{3}\right)^{15}\right)}{1 - \frac{2}{3}} = \frac{18\left(1 - \left(\frac{2}{3}\right)^{15}\right)}{\frac{1}{3}}$$

$$= 54\left(1 - \left(\tfrac{2}{3}\right)^{15}\right)$$

$$\left(\tfrac{2}{3}\right)^{15} = 0.001676\ldots$$

$$S_{15} = 54 \times 0.998324\ldots = 53.87668\ldots$$

$S_{15} = 53.877$ (to 3 d.p.)

Question 4

Worked Solution

(a) Show that $r = \frac{5}{6}$

Sum to infinity equals 6 times the first term:

$$S_\infty = 6a \implies \frac{a}{1-r} = 6a$$

Dividing both sides by a (since $a \neq 0$):

$$\frac{1}{1-r} = 6 \implies 1 = 6(1-r) \implies 1 = 6 - 6r \implies 6r = 5$$

$r = \dfrac{5}{6}$ ✓

(b) Find the value of a

Fourth term $= ar^3 = 62.5$:

$$a\left(\frac{5}{6}\right)^3 = 62.5 \implies a \times \frac{125}{216} = 62.5$$

$$a = 62.5 \times \frac{216}{125} = \frac{13500}{125}$$

$a = 108$

(c) Difference between S_∞ and S_{30}

$$S_\infty = \frac{108}{1-\frac{5}{6}} = \frac{108}{\frac{1}{6}} = 648$$

$$S_{30} = \frac{108\left(1-\left(\frac{5}{6}\right)^{30}\right)}{1-\frac{5}{6}} = 648\left(1-\left(\frac{5}{6}\right)^{30}\right)$$

$$\left(\tfrac{5}{6}\right)^{30} = 0.004198\ldots$$

$$S_{30} = 648 \times 0.995802 = 645.280\ldots$$

$$S_\infty - S_{30} = 648 - 645.280 = 2.7198\ldots$$

$S_\infty - S_{30} = 2.72$ (to 3 s.f.)

Question 5

Worked Solution

(a) Show that $11k^2 - 130k + 99 = 0$

Let $a = 7k - 5$, $ar = 5k - 7$, $ar^2 = 2k + 10$. For a geometric sequence:

$$r = \frac{5k-7}{7k-5} \quad \text{and} \quad r = \frac{2k+10}{5k-7}$$

So:

$$\frac{5k-7}{7k-5} = \frac{2k+10}{5k-7}$$

$$(5k-7)^2 = (7k-5)(2k+10)$$

$$25k^2 - 70k + 49 = 14k^2 + 70k - 10k - 50$$

$$25k^2 - 70k + 49 = 14k^2 + 60k - 50$$

$$11k^2 - 130k + 99 = 0 \checkmark$$

(b) Show that $k = \frac{9}{11}$

Factorise: $(k-11)(11k-9) = 0 \implies k = 11$ or $k = \frac{9}{11}$

Since k is not an integer:

$$k = \frac{9}{11} \checkmark$$

(c)(i) Fourth term as exact fraction

With $k = \frac{9}{11}$:

$$a = 7 \times \tfrac{9}{11} - 5 = \tfrac{63}{11} - \tfrac{55}{11} = \tfrac{8}{11}$$

$$r = \frac{5 \times \frac{9}{11} - 7}{\frac{8}{11}} = \frac{\frac{45}{11} - \frac{77}{11}}{\frac{8}{11}} = \frac{-\frac{32}{11}}{\frac{8}{11}} = -4$$

Fourth term $= ar^3 = \frac{8}{11} \times (-4)^3 = \frac{8}{11} \times (-64)$:

$$\text{Fourth term} = -\frac{512}{11}$$

(c)(ii) Sum of first 10 terms

$$S_{10} = \frac{\frac{8}{11}\left(1 - (-4)^{10}\right)}{1 - (-4)} = \frac{\frac{8}{11}(1 - 1048576)}{5} = \frac{\frac{8}{11} \times (-1048575)}{5}$$

$$= \frac{-8386200}{55} \times \frac{1}{1} = \frac{8}{55} \times (-1048575)$$

$$= \frac{-8388600}{55} \div 1$$

Let us compute directly:

$$S_{10} = \frac{\frac{8}{11}(1 - (-4)^{10})}{5} = \frac{8(1 - 1048576)}{55} = \frac{8 \times (-1048575)}{55} = \frac{-8388600}{55}$$

$S_{10} = -152520$

Question 6

Worked Solution

Second term $= ar = 750$, Fifth term $= ar^4 = -6$

(a) Find the common ratio

Divide fifth by second:

$$\frac{ar^4}{ar} = r^3 = \frac{-6}{750} = -\frac{1}{125}$$

$$r = -\frac{1}{5}$$

(b) Find the first term

$$ar = 750 \implies a \times \left(-\tfrac{1}{5}\right) = 750 \implies a = 750 \times (-5)$$

$$a = -3750$$

(c) Sum to infinity

$|r| = \frac{1}{5} < 1$, so the sum to infinity exists:

$$S_\infty = \frac{-3750}{1 - (-\frac{1}{5})} = \frac{-3750}{\frac{6}{5}} = -3750 \times \frac{5}{6}$$

$$S_\infty = -3125$$

Question 7

Worked Solution

Second term $= ar = 192$, Third term $= ar^2 = 144$

(a) Common ratio

$$r = \frac{ar^2}{ar} = \frac{144}{192}$$

$$r = \frac{3}{4}$$

(b) First term

$$a \times \frac{3}{4} = 192 \implies a = 192 \times \frac{4}{3}$$

$$a = 256$$

(c) Sum to infinity

$$S_\infty = \frac{256}{1 - \frac{3}{4}} = \frac{256}{\frac{1}{4}}$$

$$S_\infty = 1024$$

(d) Smallest n for which $S_n > 1000$

$$\frac{256\left(1 - \left(\frac{3}{4}\right)^n\right)}{1 - \frac{3}{4}} > 1000 \implies 1024\left(1 - \left(\frac{3}{4}\right)^n\right) > 1000$$

$$1 - \left(\frac{3}{4}\right)^n > \frac{1000}{1024} \implies \left(\frac{3}{4}\right)^n < 1 - \frac{1000}{1024} = \frac{24}{1024} = \frac{6}{256} = \frac{3}{128}$$

Taking logarithms (note $\log(3/4) < 0$, so inequality flips):

$$n \log\left(\frac{3}{4}\right) < \log\left(\frac{3}{128}\right)$$

$$n > \frac{\log(3/128)}{\log(3/4)} = \frac{\log 3 - \log 128}{\log 3 - \log 4} = \frac{0.4771 - 2.1072}{0.4771 - 0.6021} = \frac{-1.6301}{-0.1250} = 13.04\ldots$$

Smallest value is $n = 14$

Question 8

Worked Solution

First three terms: $(k+4)$, k, $(2k-15)$ with $k>0$.

(a) Show that $k^2 - 7k - 60 = 0$

For a geometric series the ratio is constant:

$$\frac{k}{k+4} = \frac{2k-15}{k}$$

$$k^2 = (k+4)(2k-15) = 2k^2 - 15k + 8k - 60 = 2k^2 - 7k - 60$$

$$0 = k^2 - 7k - 60 \checkmark$$

(b) Hence show that $k = 12$

$$(k-12)(k+5) = 0 \implies k = 12 \text{ or } k = -5$$

Since k is a positive constant:

$k = 12$ ✓

(c) Find the common ratio

With $k = 12$: terms are 16, 12, 9.

$$r = \frac{12}{16}$$

$r = \frac{3}{4}$

(d) Sum to infinity

$$S_\infty = \frac{16}{1-\frac{3}{4}} = \frac{16}{\frac{1}{4}}$$

$S_\infty = 64$

Question 9

Worked Solution

Car purchased for £18 000. Each year the value is 80% of the previous year's value ($r = 0.8$).

(a) Show value after exactly 3 years is £9216

$$18000 \times (0.8)^3 = 18000 \times 0.512 = 9216\checkmark$$

(b) Find n when value first falls below £1000

$$18000 \times (0.8)^n < 1000$$
$$(0.8)^n < \frac{1000}{18000} = \frac{1}{18}$$

Taking logarithms (note $\log(0.8) < 0$):

$$n\log(0.8) < \log\left(\tfrac{1}{18}\right) \implies n > \frac{\log(1/18)}{\log(0.8)} = \frac{-1.25527}{-0.09691} = 12.952\ldots$$

$n = 13$

(c) Cost of scheme in 5th year

Insurance: $a = £200$, $r = 1.12$.

$$u_5 = 200 \times (1.12)^4 = 200 \times 1.57351936 = 314.703872$$

Cost in 5th year = £314.70 (to nearest penny)

(d) Total cost of scheme for first 15 years

$$S_{15} = \frac{200\left(1.12^{15} - 1\right)}{1.12 - 1} = \frac{200(1.12^{15} - 1)}{0.12}$$
$$1.12^{15} = 5.47356\ldots$$
$$S_{15} = \frac{200 \times 4.47356}{0.12} = \frac{894.713}{0.12} = 7455.94\ldots$$

Total cost ≈ £7460 (to nearest £10)

Question 10

Worked Solution

Profit in 2006 (Year 1) = £50 000, common ratio $r > 1$.

(a) Expression for profit in Year n

Profit in Year $n = 50000\, r^{n-1}$

(b) Show that $n > \dfrac{\log 4}{\log r} + 1$

$$50000\, r^{n-1} > 200000 \implies r^{n-1} > 4$$

Taking logarithms (with $r > 1$ so $\log r > 0$):

$$(n-1)\log r > \log 4$$

$$n - 1 > \frac{\log 4}{\log r}$$

$n > \dfrac{\log 4}{\log r} + 1$ ✓

(c) With $r = 1.09$, find the year profit first exceeds £200 000

$$n > \frac{\log 4}{\log 1.09} + 1 = \frac{0.60206}{0.03743} + 1 = 16.086\ldots + 1 = 17.086\ldots \implies n = 18$$

Year 18 corresponds to $2006 + 17 =$

2023

(d) Total profits 2006–2015 (10 years)

$$S_{10} = \frac{50000(1.09^{10} - 1)}{1.09 - 1} = \frac{50000(1.09^{10} - 1)}{0.09}$$

$$1.09^{10} = 2.36736\ldots$$

$$S_{10} = \frac{50000 \times 1.36736}{0.09} = \frac{68368}{0.09} = 759646\ldots$$

Total profits ≈ £760 000 (to nearest £10 000)

Question 11

Worked Solution

First three terms: $4p$, $(3p+15)$, $(5p+20)$ with $p>0$.

(a) Show that $11p^2-10p-225=0$

$$\frac{3p+15}{4p}=\frac{5p+20}{3p+15}$$
$$(3p+15)^2=4p(5p+20)$$
$$9p^2+90p+225=20p^2+80p$$
$$0=11p^2-10p-225\checkmark$$

(b) Hence show that $p=5$

$$(p-5)(11p+45)=0 \implies p=5 \text{ or } p=-\tfrac{45}{11}$$

Since p is a positive constant:

$p=5$ ✓

(c) Find the common ratio

With $p=5$: terms are 20, 30, 45.

$$r=\frac{30}{20}$$

$r=\frac{3}{2}$

(d) Sum of first 10 terms (to nearest integer)

$$S_{10}=\frac{20\left(\left(\frac{3}{2}\right)^{10}-1\right)}{\frac{3}{2}-1}=\frac{20\left(\left(\frac{3}{2}\right)^{10}-1\right)}{\frac{1}{2}}=40\left(\left(\tfrac{3}{2}\right)^{10}-1\right)$$
$$\left(\tfrac{3}{2}\right)^{10}=\frac{3^{10}}{2^{10}}=\frac{59049}{1024}=57.6650\ldots$$
$$S_{10}=40\times(57.6650-1)=40\times 56.6650=2266.60\ldots$$

$S_{10}=2267$ (to nearest integer)

End of Worked Solutions