

Question 1 (Jun 2015, Q4)

Worked Solution

(i) Find the first three terms of the binomial expansion of $(8 - 9x)^{2/3}$ in ascending powers of x .

Take out $8^{2/3}$:

$$(8 - 9x)^{2/3} = 8^{2/3} \left(1 - \frac{9x}{8}\right)^{2/3} = 4 \left(1 - \frac{9x}{8}\right)^{2/3}.$$

Expand $\left(1 - \frac{9x}{8}\right)^{2/3}$ with $n = \frac{2}{3}$, $u = -\frac{9x}{8}$:

$$= 1 + \frac{2}{3} \left(-\frac{9x}{8}\right) + \frac{\frac{2}{3} \cdot \left(-\frac{1}{3}\right)}{2!} \left(-\frac{9x}{8}\right)^2 + \dots = 1 - \frac{3x}{4} + \frac{-\frac{2}{9}}{2} \cdot \frac{81x^2}{64} + \dots = 1 - \frac{3x}{4} - \frac{9x^2}{64} + \dots$$

Multiply by 4:

$$(8 - 9x)^{2/3} = 4 - 3x - \frac{9}{16}x^2 + \dots$$

(ii) State the set of values of x for which this expansion is valid.

The expansion is valid when $\left|\frac{9x}{8}\right| < 1$, i.e. $|x| < \frac{8}{9}$.

$$-\frac{8}{9} < x < \frac{8}{9}$$

Question 2 (Jun 2016, Q7)

Worked Solution

$(1 + kx)^n = 1 - 6x + 30x^2 + \dots$ Find n , k , and state the validity.

The general expansion: $(1 + kx)^n = 1 + nkx + \frac{n(n-1)}{2!}(kx)^2 + \dots$

From the x coefficient: $nk = -6$.

From the x^2 coefficient: $\frac{n(n-1)}{2}k^2 = 30$.

Divide the second equation by the first: $\frac{(n-1)k}{2} = \frac{30}{-6} = -5$.

So $(n-1)k = -10$, and $nk = -6$.

Subtracting: $(n-1)k - nk = -10 - (-6) \implies -k = -4 \implies k = 4$.

Then $nk = -6 \implies n = \frac{-6}{4} = -\frac{3}{2}$.

Check x^2 : $\frac{(-\frac{3}{2})(-\frac{5}{2})}{2}(4)^2 = \frac{15}{4} \cdot 16 = \frac{15}{8} \cdot 16 = 30$. ✓

The expansion is valid when $|kx| < 1$, i.e. $|4x| < 1$, i.e. $|x| < \frac{1}{4}$.

$$n = -\frac{3}{2}, k = 4; \quad \text{valid for } |x| < \frac{1}{4}$$

Question 3 (Jun 2014, Q3)

Worked Solution

(i) Find the first three terms of $(1 - 2x)^{-1/2}$, $|x| < \frac{1}{2}$.

Expand with $n = -\frac{1}{2}$, $u = -2x$:

$$(1-2x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-2x)^2 + \dots = 1 + x + \frac{3}{2}x^2 + \dots = 1 + x + \frac{3}{2}x^2 + \dots$$

$$(1 - 2x)^{-1/2} = 1 + x + \frac{3}{2}x^2 + \dots$$

(ii) Hence find the coefficient of x^2 in the expansion of $\frac{x+3}{\sqrt{1-2x}}$.

$$\frac{x+3}{\sqrt{1-2x}} = (x+3)\left(1 + x + \frac{3}{2}x^2 + \dots\right).$$

The x^2 term: $x \cdot x + 3 \cdot \frac{3}{2}x^2 = x^2 + \frac{9}{2}x^2 = \frac{11}{2}x^2$.

$$\text{Coefficient of } x^2 \text{ is } \frac{11}{2} \text{ (i.e. 5.5)}$$

Question 4 (Jun 2013, Q10)

Worked Solution

(i) Show that $\frac{x}{(1-x)^3} \approx x + 3x^2 + 6x^3$ for small x .

Expand $(1-x)^{-3}$ with $n = -3$, $u = -x$:

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \dots = 1 + 3x + 6x^2 + \dots$$

Multiply by x :

$$\frac{x}{(1-x)^3} = x(1 + 3x + 6x^2 + \dots) = x + 3x^2 + 6x^3 + \dots \quad \square$$

(ii) Use $x = 0.1$ to obtain a decimal estimate of $\frac{100}{729}$.

Note: $\frac{x}{(1-x)^3}$ at $x = 0.1$ gives $\frac{0.1}{(0.9)^3} = \frac{0.1}{0.729} = \frac{1}{7.29} = \frac{100}{729}$.

$$\frac{100}{729} \approx 0.1 + 3(0.1)^2 + 6(0.1)^3 = 0.1 + 0.03 + 0.006 = 0.136.$$

$$\frac{100}{729} \approx 0.136$$

(iii) Show $\frac{x}{(1-x)^3} = -\frac{1}{x^2} \left(1 - \frac{1}{x}\right)^{-3}$ and find the first three terms in powers of $\frac{1}{x}$.

Note that $1-x = x\left(\frac{1}{x} - 1\right) = -x\left(1 - \frac{1}{x}\right)$, so:

$$(1-x)^3 = -x^3 \left(1 - \frac{1}{x}\right)^3.$$

$$\frac{x}{(1-x)^3} = \frac{x}{-x^3 \left(1 - \frac{1}{x}\right)^3} = -\frac{1}{x^2} \left(1 - \frac{1}{x}\right)^{-3}. \quad \square$$

Expand $\left(1 - \frac{1}{x}\right)^{-3}$ with $u = -\frac{1}{x}$:

$$= 1 + (-3)\left(-\frac{1}{x}\right) + \frac{(-3)(-4)}{2!}\left(-\frac{1}{x}\right)^2 + \dots = 1 + \frac{3}{x} + \frac{6}{x^2} + \dots$$

Multiply by $-\frac{1}{x^2}$:

$$\frac{x}{(1-x)^3} = -\frac{1}{x^2} - \frac{3}{x^3} - \frac{6}{x^4} - \dots$$

First three terms in powers of $\frac{1}{x}$: $-\frac{1}{x^2} - \frac{3}{x^3} - \frac{6}{x^4} - \dots$

(iv) Comment on using $x = 0.1$ in the expansion in part (iii).

The expansion in powers of $\frac{1}{x}$ requires $\left|\frac{1}{x}\right| < 1$, i.e. $|x| > 1$. Since $x = 0.1$, we have $\left|\frac{1}{x}\right| = 10 > 1$, so the expansion is not valid and cannot be used to estimate $\frac{100}{729}$.

Not suitable: the expansion requires $|x| > 1$, but $x = 0.1 < 1$, so the series diverges.

Question 5 (Jan 2013, Q2)

Worked Solution

$(9 - 16x)^{3/2}$: find first three terms and state the validity.

$$(9 - 16x)^{3/2} = 9^{3/2} \left(1 - \frac{16x}{9}\right)^{3/2} = 27 \left(1 - \frac{16x}{9}\right)^{3/2}.$$

Expand with $n = \frac{3}{2}$, $u = -\frac{16x}{9}$:

$$\left(1 - \frac{16x}{9}\right)^{3/2} = 1 + \frac{3}{2} \left(-\frac{16x}{9}\right) + \frac{\frac{3}{2} \cdot \frac{1}{2}}{2!} \left(-\frac{16x}{9}\right)^2 + \dots = 1 - \frac{8x}{3} + \frac{3/4 \cdot 256x^2}{81} + \dots = 1 - \frac{8x}{3} + \frac{32x^2}{27} + \dots$$

Multiply by 27:

$$(9 - 16x)^{3/2} = 27 - 72x + 32x^2 + \dots$$

$$\text{Valid for } \left|\frac{16x}{9}\right| < 1, \text{ i.e. } |x| < \frac{9}{16}$$

Question 6 (Jun 2012, Q3)

Worked Solution

(i) Expand $\frac{1+x^2}{\sqrt{1+4x}}$ in ascending powers of x , up to and including the x^3 term.

Write $(1+4x)^{-1/2}$ and expand with $n = -\frac{1}{2}$, $u = 4x$:

$$\begin{aligned} (1+4x)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)(4x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(4x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(4x)^3 + \dots \\ &= 1 - 2x + \frac{3/4}{2} \cdot 16x^2 - \frac{15/8}{6} \cdot 64x^3 + \dots = 1 - 2x + 6x^2 - 20x^3 + \dots \end{aligned}$$

Multiply by $(1+x^2)$:

$$(1+x^2)(1-2x+6x^2-20x^3+\dots) = 1-2x+6x^2-20x^3+x^2-2x^3+\dots = 1-2x+7x^2-22x^3+\dots$$

$$\frac{1+x^2}{\sqrt{1+4x}} = 1 - 2x + 7x^2 - 22x^3 + \dots$$

(ii) State the set of values of x for which this expansion is valid.

Valid when $|4x| < 1$, i.e. $|x| < \frac{1}{4}$.

$$|x| < \frac{1}{4}$$

Question 7 (Jan 2012, Q4)

Worked Solution

(i) Expand $(1 - 4x)^{1/4}$ in ascending powers of x , up to and including the x^3 term.

Expand with $n = \frac{1}{4}$, $u = -4x$:

$$\begin{aligned} (1 - 4x)^{1/4} &= 1 + \frac{1}{4}(-4x) + \frac{\frac{1}{4} \cdot (-\frac{3}{4})}{2!}(-4x)^2 + \frac{\frac{1}{4} \cdot (-\frac{3}{4}) \cdot (-\frac{7}{4})}{3!}(-4x)^3 + \dots \\ &= 1 - x + \frac{-3/16}{2} \cdot 16x^2 + \frac{21/64}{6} \cdot (-64x^3) + \dots = 1 - x - \frac{3}{2}x^2 - \frac{7}{2}x^3 + \dots \end{aligned}$$

$$(1 - 4x)^{1/4} = 1 - x - \frac{3}{2}x^2 - \frac{7}{2}x^3 + \dots$$

(ii) The term of lowest degree in the expansion of $(1 + ax)(1 + bx^2)^7 - (1 - 4x)^{1/4}$ is the x^3 term. Find a and b .

Expand $(1 + ax)(1 + bx^2)^7$. Since $(1 + bx^2)^7 = 1 + 7bx^2 + \dots$, then:

$$(1 + ax)(1 + 7bx^2 + \dots) = 1 + ax + 7bx^2 + 7abx^3 + \dots$$

Subtracting $(1 - 4x)^{1/4} = 1 - x - \frac{3}{2}x^2 - \frac{7}{2}x^3 + \dots$:

$$(1 + ax + 7bx^2 + \dots) - (1 - x - \frac{3}{2}x^2 - \frac{7}{2}x^3 + \dots) = (a + 1)x + (7b + \frac{3}{2})x^2 + \dots$$

For the term of lowest degree to be x^3 , we need the x and x^2 coefficients to both equal zero:

$$x \text{ coefficient: } a + 1 = 0 \implies a = -1.$$

$$x^2 \text{ coefficient: } 7b + \frac{3}{2} = 0 \implies b = -\frac{3}{14}.$$

$$a = -1, \quad b = -\frac{3}{14}$$

End of Worked Solutions