Question 1

Worked Solution

$f(x) = (2+3x)^{-3}$, $|x| < \frac{2}{3}$. Expand up to and including the x^3 term.

Factor out 2^{-3}:

$$(2+3x)^{-3} = 2^{-3}\left(1+\frac{3x}{2}\right)^{-3} = \frac{1}{8}\left(1+\frac{3x}{2}\right)^{-3}.$$

Now expand $\left(1+\frac{3x}{2}\right)^{-3}$ using $(1+u)^n = 1+nu+\frac{n(n-1)}{2!}u^2+\frac{n(n-1)(n-2)}{3!}u^3+\cdots$ with $n=-3$, $u=\frac{3x}{2}$:

$$= 1+(-3)\left(\frac{3x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{3x}{2}\right)^3+\cdots$$

$$= 1-\frac{9x}{2}+\frac{12}{2}\cdot\frac{9x^2}{4}-\frac{60}{6}\cdot\frac{27x^3}{8}+\cdots = 1-\frac{9x}{2}+\frac{27x^2}{2}-\frac{135x^3}{4}+\cdots$$

Multiply by $\frac{1}{8}$:

$$f(x) = \frac{1}{8}-\frac{9}{16}x+\frac{27}{16}x^2-\frac{135}{32}x^3+\cdots$$

Question 2

Worked Solution

$(1 + kx)^{-4} = 1 - 6x + Ax^2 + \cdots$, $|kx| < 1$.

(a) Find k.

The general expansion: $(1 + kx)^{-4} = 1 + (-4)(kx) + \cdots = 1 - 4kx + \cdots$

Comparing the x coefficient: $-4k = -6 \implies k = \frac{3}{2}$.

$$k = \frac{3}{2}$$

(b) Find A.

The x^2 coefficient is $\frac{(-4)(-5)}{2!}(kx)^2 = \frac{20}{2}k^2 = 10k^2$.

With $k = \frac{3}{2}$:

$$A = 10 \times \left(\frac{3}{2}\right)^2 = 10 \times \frac{9}{4} = \frac{90}{4} = \frac{45}{2}.$$

$$A = \frac{45}{2}$$

Question 3

Worked Solution

$f(x) = \frac{1}{\sqrt{9+4x^2}}$, $|x| < \frac{3}{2}$. Find the first three non-zero terms.

Write as a power:

$$f(x) = (9+4x^2)^{-1/2} = \left(9\left(1+\frac{4x^2}{9}\right)\right)^{-1/2} = 9^{-1/2}\left(1+\frac{4x^2}{9}\right)^{-1/2} = \frac{1}{3}\left(1+\frac{4x^2}{9}\right)^{-1/2}.$$

Let $u = \frac{4x^2}{9}$, expand with $n = -\frac{1}{2}$:

$$\left(1+\frac{4x^2}{9}\right)^{-1/2} = 1 + (-\tfrac{1}{2})\left(\frac{4x^2}{9}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{4x^2}{9}\right)^2 + \cdots$$

$$= 1 - \frac{2x^2}{9} + \frac{\frac{3}{4}}{2}\cdot\frac{16x^4}{81} + \cdots = 1 - \frac{2x^2}{9} + \frac{2x^4}{27} + \cdots$$

Multiply by $\frac{1}{3}$:

$$f(x) = \frac{1}{3} - \frac{2}{27}x^2 + \frac{2}{81}x^4 + \cdots$$

Question 4

Worked Solution

(a) Expand $\frac{1}{(2-5x)^2}$, $|x| < \frac{2}{5}$, up to and including the x^2 term.

$$\frac{1}{(2-5x)^2} = (2-5x)^{-2} = 2^{-2}\left(1-\frac{5x}{2}\right)^{-2} = \frac{1}{4}\left(1-\frac{5x}{2}\right)^{-2}.$$

Expand with $n = -2$, $u = -\frac{5x}{2}$:

$$\left(1-\frac{5x}{2}\right)^{-2} = 1+(-2)\left(-\frac{5x}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{5x}{2}\right)^2+\cdots = 1+5x+\frac{6}{2}\cdot\frac{25x^2}{4}+\cdots = 1+5x+\frac{75x^2}{4}+\cdots$$

$$\frac{1}{(2-5x)^2} = \frac{1}{4}+\frac{5}{4}x+\frac{75}{16}x^2+\cdots$$

(b) The expansion of $\frac{2+kx}{(2-5x)^2}$ begins $\frac{1}{2}+\frac{7}{4}x+Ax^2+\cdots$ Find k.

$$\frac{2+kx}{(2-5x)^2} = (2+kx)\left(\frac{1}{4}+\frac{5}{4}x+\frac{75}{16}x^2+\cdots\right).$$

The constant term: $2\cdot\frac{1}{4} = \frac{1}{2}$. ✓

The x term: $2\cdot\frac{5}{4}x + kx\cdot\frac{1}{4} = \frac{5}{2}x+\frac{k}{4}x = \frac{7}{4}x$.

So $\frac{5}{2}+\frac{k}{4} = \frac{7}{4} \implies \frac{k}{4} = \frac{7}{4}-\frac{10}{4} = -\frac{3}{4} \implies k = -3$.

$k = -3$

(c) Find A.

The x^2 term: $2\cdot\frac{75}{16}x^2 + kx\cdot\frac{5}{4}x = \frac{150}{16}x^2+\frac{5k}{4}x^2 = \frac{75}{8}x^2+\frac{5(-3)}{4}x^2 = \frac{75}{8}x^2-\frac{15}{4}x^2$.

$$A = \frac{75}{8}-\frac{30}{8} = \frac{45}{8}.$$

$A = \frac{45}{8}$

Question 5

Worked Solution

(a) Expand $\sqrt[3]{8-9x}$, $|x| < \dfrac{8}{9}$, up to and including the x^3 term.

Write as a power with index $\dfrac{1}{3}$:

$$(8-9x)^{1/3} = 8^{1/3}\left(1-\frac{9x}{8}\right)^{1/3} = 2\left(1-\frac{9x}{8}\right)^{1/3}.$$

Expand with $n = \dfrac{1}{3}$, $u = -\dfrac{9x}{8}$:

$$\left(1-\frac{9x}{8}\right)^{1/3} = 1 + \frac{1}{3}\left(-\frac{9x}{8}\right) + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2!}\left(-\frac{9x}{8}\right)^2 + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{5}{3}\right)}{3!}\left(-\frac{9x}{8}\right)^3 + \cdots$$

$$= 1 - \frac{3x}{8} + \frac{-\frac{2}{9}}{2}\cdot\frac{81x^2}{64} - \frac{\frac{10}{27}}{6}\cdot\frac{729x^3}{512} + \cdots = 1 - \frac{3x}{8} - \frac{9x^2}{64} - \frac{45x^3}{512} + \cdots$$

Multiply by 2:

$$(8-9x)^{1/3} = 2 - \frac{3}{4}x - \frac{9}{32}x^2 - \frac{45}{256}x^3 + \cdots$$

(b) Use your expansion to estimate $\sqrt[3]{7100}$ to 4 decimal places.

Note: $7100 = 10^3 \times 7.1 = 1000 \times 7.1$. Instead, observe:

$$\sqrt[3]{7100} = \sqrt[3]{10^3 \times 7.1} = 10\sqrt[3]{7.1} = 10\sqrt[3]{8-0.9} = 10(8-9x)^{1/3} \text{ with } x = 0.1.$$

Check: $|0.1| < \dfrac{8}{9}$. ✓

Substitute $x = 0.1$ into the expansion:

$$(8-9(0.1))^{1/3} \approx 2 - \frac{3}{4}(0.1) - \frac{9}{32}(0.1)^2 - \frac{45}{256}(0.1)^3 = 2 - 0.075 - 0.0028125 - 0.000175781\ldots \approx 1.92201171$$

$$\sqrt[3]{7100} = 10 \times 1.922011719\ldots$$

$$\sqrt[3]{7100} \approx 19.2201 \text{ (to 4 d.p.)}$$

Question 6

Worked Solution

(a) Expand $\frac{1}{\sqrt{9-10x}}$, $|x| < \frac{9}{10}$, up to and including the x^2 term.

$$(9-10x)^{-1/2} = 9^{-1/2}\left(1 - \frac{10x}{9}\right)^{-1/2} = \frac{1}{3}\left(1 - \frac{10x}{9}\right)^{-1/2}.$$

Expand with $n = -\frac{1}{2}$, $u = -\frac{10x}{9}$:

$$\left(1 - \frac{10x}{9}\right)^{-1/2} = 1+(-\tfrac{1}{2})\left(-\frac{10x}{9}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{10x}{9}\right)^2+\cdots = 1+\frac{5x}{9}+\frac{3/4}{2}\cdot\frac{100x^2}{81}+\cdots = 1+\frac{5x}{9}+2$$

Multiply by $\frac{1}{3}$:

$$\frac{1}{\sqrt{9-10x}} = \frac{1}{3} + \frac{5}{27}x + \frac{25}{162}x^2 + \cdots$$

(b) Hence find the expansion of $\frac{3+x}{\sqrt{9-10x}}$ up to and including the x^2 term.

$$\frac{3+x}{\sqrt{9-10x}} = (3+x)\left(\frac{1}{3} + \frac{5}{27}x + \frac{25}{162}x^2 + \cdots\right).$$

Constant term: $3 \cdot \frac{1}{3} = 1$.

x term: $3 \cdot \frac{5}{27}x + x \cdot \frac{1}{3} = \frac{5}{9}x + \frac{1}{3}x = \frac{5}{9}x + \frac{3}{9}x = \frac{8}{9}x$.

x^2 term: $3 \cdot \frac{25}{162}x^2 + x \cdot \frac{5}{27}x = \frac{25}{54}x^2 + \frac{5}{27}x^2 = \frac{25}{54}x^2 + \frac{10}{54}x^2 = \frac{35}{54}x^2$.

$$\frac{3+x}{\sqrt{9-10x}} = 1 + \frac{8}{9}x + \frac{35}{54}x^2 + \cdots$$

Question 7

Worked Solution

(a) Show that $\sqrt{\dfrac{1+x}{1-x}} \approx 1 + x + \dfrac{1}{2}x^2$, $|x| < 1$.

Write $\sqrt{\dfrac{1+x}{1-x}} = (1+x)^{1/2}(1-x)^{-1/2}$.

Expand $(1+x)^{1/2}$ with $n = \frac{1}{2}$:

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot(-\frac{1}{2})}{2!}x^2 + \cdots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

Expand $(1-x)^{-1/2}$ with $n = -\frac{1}{2}$, $u = -x$:

$$(1-x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x)^2 + \cdots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Multiply the two expansions, keeping terms up to x^2:

$$\left(1 + \frac{x}{2} - \frac{x^2}{8} + \cdots\right)\left(1 + \frac{x}{2} + \frac{3x^2}{8} + \cdots\right)$$

$$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} + \cdots = 1 + x + \left(\frac{3}{8} + \frac{2}{8} - \frac{1}{8}\right)x^2 + \cdots = 1 + x + \frac{1}{2}x^2 + \cdots \quad \square$$

(b) Substitute $x = \dfrac{1}{26}$ to obtain an approximation to $\sqrt{3}$ in the form $\dfrac{a}{b}$.

With $x = \dfrac{1}{26}$:

$$\sqrt{\frac{1+\frac{1}{26}}{1-\frac{1}{26}}} = \sqrt{\frac{27/26}{25/26}} = \sqrt{\frac{27}{25}} = \frac{\sqrt{27}}{5} = \frac{3\sqrt{3}}{5}.$$

The approximation gives:

$$\frac{3\sqrt{3}}{5} \approx 1 + \frac{1}{26} + \frac{1}{2}\left(\frac{1}{26}\right)^2 = 1 + \frac{1}{26} + \frac{1}{1352} = \frac{1352 + 52 + 1}{1352} = \frac{1405}{1352}.$$

So $3\sqrt{3} \approx \dfrac{5 \times 1405}{1352} = \dfrac{7025}{1352}$, giving $\sqrt{3} \approx \dfrac{7025}{4056}$.

$$\sqrt{3} \approx \frac{7025}{4056}$$

Question 8

Worked Solution

(a) Expand $(8-3x)^{1/3}$, $|x| < \dfrac{8}{3}$, up to and including the x^3 term.

$$(8-3x)^{1/3} = 8^{1/3}\left(1-\frac{3x}{8}\right)^{1/3} = 2\left(1-\frac{3x}{8}\right)^{1/3}.$$

Expand with $n = \frac{1}{3}$, $u = -\frac{3x}{8}$:

$$\left(1-\frac{3x}{8}\right)^{1/3} = 1 + \frac{1}{3}\left(-\frac{3x}{8}\right) + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)}{2!}\left(-\frac{3x}{8}\right)^2 + \frac{\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{5}{3}\right)}{3!}\left(-\frac{3x}{8}\right)^3 + \cdots$$

$$= 1 - \frac{x}{8} - \frac{\frac{2}{9}}{2}\cdot\frac{9x^2}{64} - \frac{\frac{10}{27}}{6}\cdot\frac{27x^3}{512} + \cdots = 1 - \frac{x}{8} - \frac{x^2}{16} - \frac{5x^3}{384} + \cdots$$

Multiply by 2:

$$(8-3x)^{1/3} = 2 - \frac{x}{4} - \frac{x^2}{32} - \frac{5x^3}{768} + \cdots$$

(b) Use a suitable value of x to approximate $\sqrt[3]{7.7}$ to 7 decimal places.

Note: $7.7 = 8 - 0.3 = 8 - 3(0.1)$, so take $x = 0.1$:

$$\sqrt[3]{7.7} \approx 2 - \frac{0.1}{4} - \frac{(0.1)^2}{32} - \frac{5(0.1)^3}{768} = 2 - 0.025 - 0.0003125 - 0.0000065104\ldots = 1.9746810\ldots$$

$$\sqrt[3]{7.7} \approx 1.9746810$$

Question 9

Worked Solution

$\dfrac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$.

(a) Find A, B and C.

Since the degree of the numerator equals the degree of the denominator (both degree 2), A is the leading coefficient ratio.

Expanding $(x-1)(x+2) = x^2+x-2$. Doing polynomial division: $2x^2+5x-10 = 2(x^2+x-2)+(3x-6)$.

So $\dfrac{2x^2+5x-10}{(x-1)(x+2)} = 2 + \dfrac{3x-6}{(x-1)(x+2)}$, giving $A=2$.

Now write $\dfrac{3x-6}{(x-1)(x+2)} = \dfrac{B}{x-1} + \dfrac{C}{x+2}$:

$3x-6 \equiv B(x+2) + C(x-1)$.

$x=1$: $3-6=3B \implies B=-1$.

$x=-2$: $-6-6=-3C \implies C=4$.

$A=2,\ B=-1,\ C=4$

(b) Expand $\dfrac{2x^2+5x-10}{(x-1)(x+2)}$ in ascending powers of x, up to the x^2 term.

Using the partial fractions: $\dfrac{2x^2+5x-10}{(x-1)(x+2)} = 2-(1-x)^{-1}+2\left(1+\dfrac{x}{2}\right)^{-1}$ ...

More carefully: $\dfrac{B}{x-1} = \dfrac{-1}{x-1} = \dfrac{1}{1-x} = (1-x)^{-1}$ and $\dfrac{C}{x+2} = \dfrac{4}{x+2} = \dfrac{4}{2(1+x/2)} = \dfrac{2}{1+x/2} = 2\left(1+\dfrac{x}{2}\right)^{-1}$.

Expand:

$$(1-x)^{-1} = 1+x+x^2+\cdots$$

$$2\left(1+\frac{x}{2}\right)^{-1} = 2\left(1-\frac{x}{2}+\frac{x^2}{4}-\cdots\right) = 2-x+\frac{x^2}{2}-\cdots$$

So:

$$\frac{2x^2+5x-10}{(x-1)(x+2)} = 2+(1+x+x^2+\cdots)+\left(2-x+\frac{x^2}{2}+\cdots\right) = 5+0\cdot x+\frac{3}{2}x^2+\cdots$$

$$\frac{2x^2+5x-10}{(x-1)(x+2)} = 5+\frac{3}{2}x^2+\cdots$$

(The x term vanishes since $1-1=0$.)

Question 10

Worked Solution

$\mathrm{f}(x) = \dfrac{50x^2 + 38x + 9}{(5x+2)^2(1-2x)}$, $x \neq -\dfrac{2}{5}$, $x \neq \dfrac{1}{2}$.

$\mathrm{f}(x) \equiv \dfrac{A}{5x+2} + \dfrac{B}{(5x+2)^2} + \dfrac{C}{1-2x}$.

(a)(i) Find B and C; **(ii)** Show $A = 0$.

Multiply both sides by $(5x+2)^2(1-2x)$:

$$50x^2 + 38x + 9 \equiv A(5x+2)(1-2x) + B(1-2x) + C(5x+2)^2.$$

$x = -\dfrac{2}{5}$: $\quad 50 \cdot \dfrac{4}{25} + 38 \cdot (-\dfrac{2}{5}) + 9 = B\left(1 + \dfrac{4}{5}\right)$.

LHS: $8 - \dfrac{76}{5} + 9 = 17 - 15.2 = 1.8 = \dfrac{9}{5}$. RHS: $B \cdot \dfrac{9}{5}$.

So $B = 1$.

$x = \dfrac{1}{2}$: $\quad 50 \cdot \dfrac{1}{4} + 38 \cdot \dfrac{1}{2} + 9 = C\left(5 \cdot \dfrac{1}{2} + 2\right)^2$.

LHS: $12.5 + 19 + 9 = 40.5 = \dfrac{81}{2}$. RHS: $C \cdot \left(\dfrac{9}{2}\right)^2 = C \cdot \dfrac{81}{4}$.

So $C = \dfrac{81/2}{81/4} = 2$.

To show $A = 0$: substitute $x = 0$:

$$9 = A(2)(1) + B(1) + C(4) = 2A + 1 + 8 = 2A + 9 \implies 2A = 0 \implies A = 0. \quad \square$$

$$\boxed{B = 1,\ C = 2,\ A = 0}$$

(b)(i) Use binomial expansions to show $\mathrm{f}(x) = p + qx + rx^2 + \cdots$ with p, q, r simplified fractions.

With $A = 0$:

$$\mathrm{f}(x) = \frac{1}{(5x+2)^2} + \frac{2}{1-2x}.$$

$$\frac{1}{(5x+2)^2} = (5x+2)^{-2} = 2^{-2}\left(1 + \frac{5x}{2}\right)^{-2} = \frac{1}{4}\left(1 + \frac{5x}{2}\right)^{-2}.$$

Expand with $n = -2$:

$$\frac{1}{4}\left[1 + (-2)\left(\frac{5x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{5x}{2}\right)^2 + \cdots\right] = \frac{1}{4}\left[1 - 5x + \frac{75x^2}{4} + \cdots\right] = \frac{1}{4} - \frac{5x}{4} + \frac{75x^2}{16} + \cdots$$

$\frac{2}{1-2x} = 2(1-2x)^{-1} = 2(1+2x+4x^2+\cdots) = 2+4x+8x^2+\cdots$

Adding:

$$f(x) = \left(\frac{1}{4}+2\right) + \left(-\frac{5}{4}+4\right)x + \left(\frac{75}{16}+8\right)x^2 + \cdots = \frac{9}{4} + \frac{11}{4}x + \frac{203}{16}x^2 + \cdots$$

$$p = \frac{9}{4}, \quad q = \frac{11}{4}, \quad r = \frac{203}{16}$$

(b)(ii) Find the range of values of x for which the expansion is valid.

The expansion of $(1+\frac{5x}{2})^{-2}$ is valid for $\left|\frac{5x}{2}\right| < 1$, i.e. $|x| < \frac{2}{5}$.

The expansion of $(1-2x)^{-1}$ is valid for $|2x| < 1$, i.e. $|x| < \frac{1}{2}$.

The expansion is valid where both hold, i.e. the stricter condition:

$$|x| < \frac{2}{5}$$

End of Worked Solutions