Question 1

Worked Solution

(a) We need to show that $f(x) = \frac{1}{x+1}$, $x > 3$.

Start with

$$f(x) = \frac{2(x-1)}{x^2 - 2x - 3} - \frac{1}{x-3}.$$

Factorise the denominator of the first fraction:

$$x^2 - 2x - 3 = (x-3)(x+1).$$

So

$$f(x) = \frac{2(x-1)}{(x-3)(x+1)} - \frac{1}{x-3}.$$

Write both fractions over the common denominator $(x-3)(x+1)$:

$$f(x) = \frac{2(x-1)}{(x-3)(x+1)} - \frac{x+1}{(x-3)(x+1)} = \frac{2(x-1)-(x+1)}{(x-3)(x+1)}.$$

Expand the numerator:

$$2(x-1)-(x+1) = 2x - 2 - x - 1 = x - 3.$$

Therefore

$$f(x) = \frac{x-3}{(x-3)(x+1)} = \frac{1}{x+1}, \quad x > 3. \qquad \square$$

(b) Find the range of f.

Since $x > 3$, we have $x + 1 > 4$, so $\frac{1}{x+1} < \frac{1}{4}$.

Also $f(x) = \frac{1}{x+1} > 0$ for all $x > 3$.

As $x \to \infty$, $f(x) \to 0^+$; as $x \to 3^+$, $f(x) \to \frac{1}{4}^-$.

Range of f: $\quad 0 < f(x) < \frac{1}{4}$

(c) Find $f^{-1}(x)$ and state its domain.

Let $y = \frac{1}{x+1}$. Rearrange to make x the subject:

$$x + 1 = \frac{1}{y} \implies x = \frac{1}{y} - 1 = \frac{1-y}{y}.$$

Replace y with x:

$\mathrm{f}^{-1}(x) = \dfrac{1-x}{x} = \dfrac{1}{x} - 1$

Domain of f^{-1}: $0 < x < \dfrac{1}{4}$ (equals the range of f)

(d) Solve $\mathrm{fg}(x) = \frac{1}{8}$, where $\mathrm{g}(x) = 2x^2 - 3$.

First apply f to $\mathrm{g}(x)$. Since $\mathrm{f}(t) = \dfrac{1}{t+1}$:

$$\mathrm{fg}(x) = \frac{1}{(2x^2-3)+1} = \frac{1}{2x^2-2}.$$

Set equal to $\frac{1}{8}$:

$$\frac{1}{2x^2-2} = \frac{1}{8} \implies 2x^2 - 2 = 8 \implies 2x^2 = 10 \implies x^2 = 5.$$

$x = \pm\sqrt{5}$

Question 2

Worked Solution

$\mathrm{f} : x \mapsto \dfrac{3(x+1)}{2x^2+7x-4} - \dfrac{1}{x+4}, \quad x \in \mathbb{R},\ x > \dfrac{1}{2}.$

(a) Show that $\mathrm{f}(x) = \dfrac{1}{2x-1}$.

Factorise the denominator of the first fraction:

$$2x^2 + 7x - 4 = (2x-1)(x+4).$$

So

$$\mathrm{f}(x) = \frac{3(x+1)}{(2x-1)(x+4)} - \frac{1}{x+4}.$$

Common denominator $(2x-1)(x+4)$:

$$\mathrm{f}(x) = \frac{3(x+1)-(2x-1)}{(2x-1)(x+4)} = \frac{3x+3-2x+1}{(2x-1)(x+4)} = \frac{x+4}{(2x-1)(x+4)} = \frac{1}{2x-1}. \quad \square$$

(b) Find $\mathrm{f}^{-1}(x)$.

Let $y = \dfrac{1}{2x-1}$. Rearrange:

$$2x - 1 = \frac{1}{y} \implies 2x = \frac{1}{y} + 1 = \frac{1+y}{y} \implies x = \frac{1+y}{2y}.$$

$$\mathrm{f}^{-1}(x) = \frac{1+x}{2x}$$

(c) Find the domain of f^{-1}.

The domain of f^{-1} equals the range of f.

Since $x > \dfrac{1}{2}$, we have $2x - 1 > 0$, so $\mathrm{f}(x) = \dfrac{1}{2x-1} > 0$.

As $x \to \frac{1}{2}^{+}$, $\mathrm{f}(x) \to +\infty$; as $x \to \infty$, $\mathrm{f}(x) \to 0^{+}$.

Domain of f^{-1}: $x > 0$

(d) $\mathrm{g}(x) = \ln(x+1)$. Solve $\mathrm{fg}(x) = \dfrac{1}{7}$, giving the answer in terms of e.

$$\mathrm{fg}(x) = \mathrm{f}(\ln(x+1)) = \frac{1}{2\ln(x+1) - 1}.$$

Set equal to $\frac{1}{7}$:

$$\frac{1}{2\ln(x+1)-1} = \frac{1}{7} \implies 2\ln(x+1) - 1 = 7 \implies 2\ln(x+1) = 8 \implies \ln(x+1) = 4.$$

Exponentiate:

$$x + 1 = e^4 \implies x = e^4 - 1.$$

$x = e^4 - 1$

Question 3

Worked Solution

$\mathrm{f} : x \mapsto 1 - 2x^3,\ x \in \mathbb{R}; \quad \mathrm{g} : x \mapsto \dfrac{3}{x} - 4,\ x > 0.$

(a) Find f^{-1}.

Let $y = 1 - 2x^3$:

$$2x^3 = 1 - y \implies x^3 = \frac{1-y}{2} \implies x = \left(\frac{1-y}{2}\right)^{1/3}.$$

$$\mathrm{f}^{-1}(x) = \left(\frac{1-x}{2}\right)^{1/3}$$

(b) Show that $\mathrm{gf} : x \mapsto \dfrac{8x^3 - 1}{1 - 2x^3}$.

$$\mathrm{gf}(x) = \mathrm{g}\big(1 - 2x^3\big) = \frac{3}{1-2x^3} - 4 = \frac{3 - 4(1-2x^3)}{1-2x^3} = \frac{3 - 4 + 8x^3}{1-2x^3} = \frac{8x^3 - 1}{1-2x^3}. \quad \square$$

(c) Solve $\mathrm{gf}(x) = 0$.

$$\frac{8x^3-1}{1-2x^3} = 0 \implies 8x^3 - 1 = 0 \implies x^3 = \frac{1}{8} \implies x = \frac{1}{2}.$$

(Note: the denominator $1 - 2x^3 \neq 0$ when $x = \frac{1}{2}$ since $1 - 2 \cdot \frac{1}{8} = \frac{3}{4} \neq 0$.)

$$x = \frac{1}{2}$$

(d) Use calculus to find the coordinates of the stationary point on $y = \mathrm{gf}(x)$.

Write $y = \dfrac{8x^3-1}{1-2x^3}$. Use the quotient rule with $u = 8x^3 - 1$, $v = 1 - 2x^3$:

$$u' = 24x^2, \quad v' = -6x^2.$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{24x^2(1-2x^3) - (8x^3-1)(-6x^2)}{(1-2x^3)^2}.$$

Expand the numerator:

$$24x^2 - 48x^5 + 6x^2(8x^3 - 1) = 24x^2 - 48x^5 + 48x^5 - 6x^2 = 18x^2.$$

So $\dfrac{dy}{dx} = \dfrac{18x^2}{(1-2x^3)^2}$.

Set $\dfrac{dy}{dx} = 0$: numerator $= 0 \implies 18x^2 = 0 \implies x = 0$.

When $x = 0$: $y = \dfrac{0-1}{1-0} = -1$.

Stationary point at $(0, -1)$

Question 4

Worked Solution

$\mathrm{f} : x \mapsto \ln(2x-1),\ x \in \mathbb{R},\ x > \frac{1}{2};\quad \mathrm{g} : x \mapsto \dfrac{2}{x-3},\ x \in \mathbb{R},\ x \neq 3.$

(a) Find the exact value of fg(4).

$$\mathrm{g}(4) = \frac{2}{4-3} = 2.$$
$$\mathrm{fg}(4) = \mathrm{f}(2) = \ln(2 \cdot 2 - 1) = \ln 3.$$

$\mathrm{fg}(4) = \ln 3$

(b) Find $\mathrm{f}^{-1}(x)$, stating its domain.

Let $y = \ln(2x-1)$. Exponentiate:

$$e^y = 2x - 1 \implies 2x = e^y + 1 \implies x = \frac{e^y+1}{2}.$$

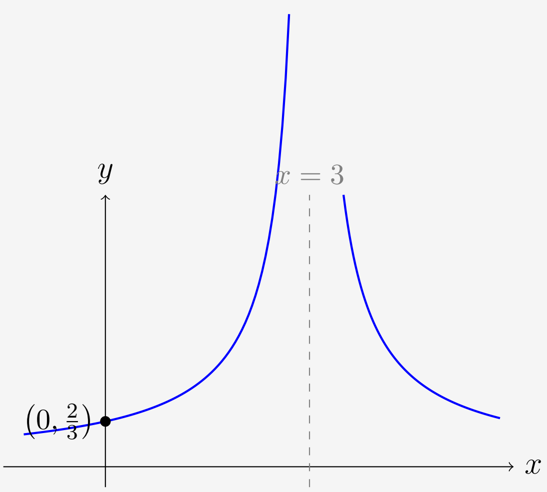
$\mathrm{f}^{-1}(x) = \dfrac{e^x+1}{2}$, domain: $x \in \mathbb{R}$ (all real numbers)

(The domain of f^{-1} equals the range of f; since f has domain $x > \frac{1}{2}$, its range is all of $\mathbb{R}$.)

(c) Sketch $y = |\mathrm{g}(x)|$.

$\mathrm{g}(x) = \dfrac{2}{x-3}$: vertical asymptote $x = 3$, crosses y-axis at $\mathrm{g}(0) = \dfrac{2}{0-3} = -\dfrac{2}{3}$.

For $|\mathrm{g}(x)|$: the part of the curve below the x-axis is reflected upward. The vertical asymptote remains $x = 3$; there is no horizontal asymptote crossing. The graph crosses the y-axis at $\left(0, \dfrac{2}{3}\right)$.



(d) Find the exact values of x for which $\left|\dfrac{2}{x-3}\right| = 3$.

This gives $\dfrac{2}{x-3} = 3$ or $\dfrac{2}{x-3} = -3$.

Case 1: $\dfrac{2}{x-3} = 3 \implies x - 3 = \dfrac{2}{3} \implies x = 3 + \dfrac{2}{3} = \dfrac{11}{3}$.

Case 2: $\dfrac{2}{x-3} = -3 \implies x - 3 = -\dfrac{2}{3} \implies x = 3 - \dfrac{2}{3} = \dfrac{7}{3}$.

$$x = \frac{7}{3} \quad \text{or} \quad x = \frac{11}{3}$$

Question 5

Worked Solution

$\text{f} : x \to \dfrac{3-2x}{x-5}$, $x \in \mathbb{R}$, $x \neq 5$.

(a) Find $\text{f}^{-1}(x)$.

Let $y = \dfrac{3-2x}{x-5}$. Rearrange:

$y(x-5) = 3-2x \implies xy-5y = 3-2x \implies xy+2x = 3+5y \implies x(y+2) = 3+5y.$

$$x = \frac{3+5y}{y+2}.$$

$\text{f}^{-1}(x) = \dfrac{5x+3}{x+2}, \quad x \neq -2$

The function g is linear from $(-1,-9)$ to $(2,0)$ and from $(2,0)$ to $(8,4)$.

(b) Range of g:

The outputs run from -9 (at $x=-1$) to 4 (at $x=8$), passing through 0 (at $x=2$).

Range of g: $-9 \leq y \leq 4$

(c) Find gg(2).

$\text{g}(2) = 0$ (given). Then $\text{gg}(2) = \text{g}(0)$.

On the segment from $(-1,-9)$ to $(2,0)$: gradient $= \dfrac{0-(-9)}{2-(-1)} = \dfrac{9}{3} = 3$.

So $\text{g}(x) = 3(x+1) - 9 = 3x - 6$ on $-1 \leq x \leq 2$.

$\text{g}(0) = 3(0) - 6 = -6$.

$\text{gg}(2) = -6$

(d) Find fg(8).

$\text{g}(8) = 4$ (given endpoint). Then:

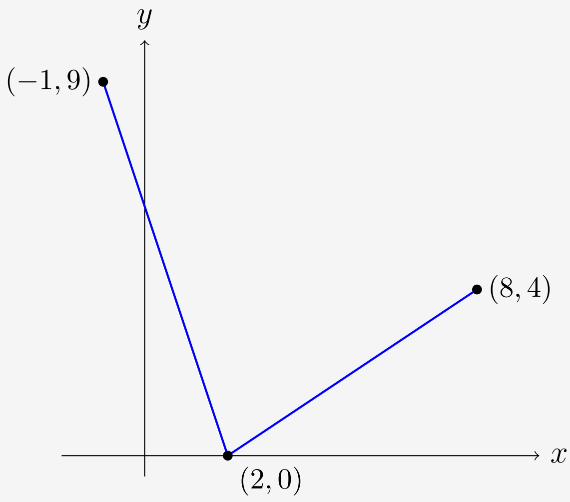
$$\text{fg}(8) = \text{f}(4) = \frac{3-2(4)}{4-5} = \frac{3-8}{-1} = \frac{-5}{-1} = 5.$$

$\text{fg}(8) = 5$

(e) Sketches:

(i) $y = |\text{g}(x)|$: reflect the part from $(-1,-9)$ to $(2,0)$ in the x-axis (since $\text{g} \leq 0$ there); the segment from $(2,0)$ to $(8,4)$ stays unchanged.

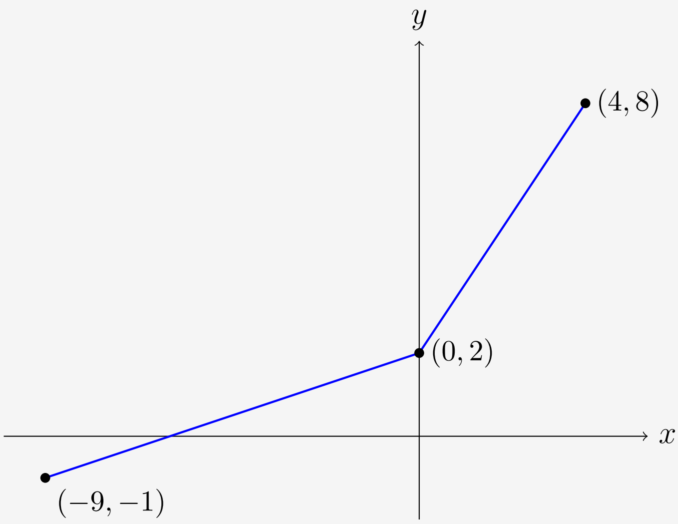
Key coordinates: $(-1,9)$, $(2,0)$, $(8,4)$. The graph meets the x-axis at $(2,0)$.



(ii) $y = \mathrm{g}^{-1}(x)$: reflect the original g in the line $y = x$.

The points $(-1, -9)$, $(2, 0)$, $(8, 4)$ become $(-9, -1)$, $(0, 2)$, $(4, 8)$.

The inverse is two linear segments from $(-9, -1)$ to $(0, 2)$ and from $(0, 2)$ to $(4, 8)$.



(f) Domain of g^{-1}:

The domain of g^{-1} equals the range of g.

Domain of g^{-1}: $-9 \le x \le 4$

Question 6

Worked Solution

$f : x \mapsto 2|x| + 3,\ x \in \mathbb{R};\quad g : x \mapsto 3 - 4x,\ x \in \mathbb{R}.$

(a) State the range of f.

Since $|x| \geq 0$ for all x, we have $2|x| + 3 \geq 3$.

Range of f: $f(x) \geq 3$

(b) Find fg(1).

$$g(1) = 3 - 4(1) = -1.$$
$$fg(1) = f(-1) = 2|-1| + 3 = 2(1) + 3 = 5.$$

$fg(1) = 5$

(c) Find $g^{-1}(x)$.

Let $y = 3 - 4x$:

$$4x = 3 - y \implies x = \frac{3 - y}{4}.$$

$g^{-1}(x) = \dfrac{3 - x}{4}$

(d) Solve $gg(x) + [g(x)]^2 = 0$.

First compute $g(x) = 3 - 4x$, then $gg(x) = g(3 - 4x) = 3 - 4(3 - 4x) = 3 - 12 + 16x = 16x - 9$.

Let $u = g(x) = 3 - 4x$. The equation becomes:

$$gg(x) + [g(x)]^2 = (16x - 9) + (3 - 4x)^2 = 0.$$

Expand $(3 - 4x)^2 = 9 - 24x + 16x^2$:

$$16x - 9 + 9 - 24x + 16x^2 = 0 \implies 16x^2 - 8x = 0 \implies 8x(2x - 1) = 0.$$

$x = 0 \quad \text{or} \quad x = \dfrac{1}{2}$

Question 7

Worked Solution

$g(x) = 3 + \sqrt{x+2}, \quad x \geq -2.$

(a) State the range of g.

Since $x \geq -2$, we have $\sqrt{x+2} \geq 0$, so $g(x) \geq 3$.

Range of g: $g(x) \geq 3$

(b) Find $g^{-1}(x)$ and state its domain.

Let $y = 3 + \sqrt{x+2}$:

$$\sqrt{x+2} = y - 3 \implies x + 2 = (y-3)^2 \implies x = (y-3)^2 - 2.$$

$g^{-1}(x) = (x-3)^2 - 2, \quad$ domain: $x \geq 3$

(c) Find the exact value of x for which $g(x) = x$.

$$3 + \sqrt{x+2} = x \implies \sqrt{x+2} = x - 3.$$

For this to be valid we need $x \geq 3$. Square both sides:

$$x + 2 = (x-3)^2 = x^2 - 6x + 9 \implies x^2 - 7x + 7 = 0.$$

Using the quadratic formula:

$$x = \frac{7 \pm \sqrt{49-28}}{2} = \frac{7 \pm \sqrt{21}}{2}.$$

Since we need $x \geq 3$: check $\dfrac{7-\sqrt{21}}{2} \approx \dfrac{7-4.58}{2} \approx 1.21 < 3$ (rejected).

$x = \dfrac{7+\sqrt{21}}{2}$

(d) Hence state the value of a for which $g(a) = g^{-1}(a)$.

When $g(x) = x$, we also have $g^{-1}(x) = x$ (since if $g(a) = a$ then applying g^{-1} gives $a = g^{-1}(a)$). Therefore:

$a = \dfrac{7+\sqrt{21}}{2}$

Question 8

Worked Solution

$f : x \to e^{2x} + k^2$, $x \in \mathbb{R}$, k a positive constant.

(a) State the range of f.

Since $e^{2x} > 0$ for all x, we have $e^{2x} + k^2 > k^2$.

Range of f: $f(x) > k^2$

(b) Find f^{-1} and state its domain.

Let $y = e^{2x} + k^2$:

$$e^{2x} = y - k^2 \implies 2x = \ln(y - k^2) \implies x = \frac{1}{2}\ln(y - k^2).$$

$f^{-1}(x) = \frac{1}{2}\ln(x - k^2)$, domain: $x > k^2$

(c) $g(x) = \ln(2x)$, $x > 0$. Solve $g(x) + g(x^2) + g(x^3) = 6$.

Using log laws:

$$\ln(2x) + \ln(2x^2) + \ln(2x^3) = 6.$$

$$\ln\left(2x \cdot 2x^2 \cdot 2x^3\right) = 6 \implies \ln(8x^6) = 6 \implies 8x^6 = e^6.$$

$$x^6 = \frac{e^6}{8} \implies x = \left(\frac{e^6}{8}\right)^{1/6} = \frac{e}{8^{1/6}} = \frac{e}{2^{1/2}} = \frac{e}{\sqrt{2}}.$$

$x = \frac{e}{\sqrt{2}}$

(d) Find $fg(x)$ in simplest form.

$$fg(x) = f(\ln(2x)) = e^{2\ln(2x)} + k^2 = (2x)^2 + k^2 = 4x^2 + k^2.$$

$fg(x) = 4x^2 + k^2$

(e) Solve $fg(x) = 2k^2$.

$$4x^2 + k^2 = 2k^2 \implies 4x^2 = k^2 \implies x^2 = \frac{k^2}{4} \implies x = \frac{k}{2}.$$

(Taking positive root since $x > 0$ is the domain of g, and $k > 0$.)

$x = \frac{k}{2}$

Question 9

Worked Solution

The function f has domain $-2 \leq x \leq 6$, linear from $(-2, 10)$ to $(2, 0)$ and from $(2, 0)$ to $(6, 4)$.

(a) Write down the range of f.

The function takes all values from 0 (minimum at $x = 2$) to 10 (at $x = -2$), passing through 4 (at $x = 6$).

Range of f: $0 \leq \mathrm{f}(x) \leq 10$

(b) Find ff(0).

On segment $(-2, 10)$ to $(2, 0)$: gradient $= \dfrac{0-10}{2-(-2)} = \dfrac{-10}{4} = -\dfrac{5}{2}$.

Using point $(2, 0)$: $\mathrm{f}(x) = -\dfrac{5}{2}(x-2)$ for $-2 \leq x \leq 2$.

$\mathrm{f}(0) = -\dfrac{5}{2}(0-2) = 5$.

Now find f(5). On segment $(2, 0)$ to $(6, 4)$: gradient $= \dfrac{4-0}{6-2} = 1$.

$\mathrm{f}(x) = x - 2$ for $2 \leq x \leq 6$.

$\mathrm{f}(5) = 5 - 2 = 3$.

$\mathrm{ff}(0) = 3$

$\mathrm{g} : x \to \dfrac{4+3x}{5-x},\ x \in \mathbb{R},\ x \neq 5.$

(c) Find $\mathrm{g}^{-1}(x)$.

Let $y = \dfrac{4+3x}{5-x}$:

$$y(5-x) = 4+3x \implies 5y-xy = 4+3x \implies 5y-4 = xy+3x = x(y+3) \implies x = \frac{5y-4}{y+3}.$$

$\mathrm{g}^{-1}(x) = \dfrac{5x-4}{x+3}, \quad x \neq -3$

(d) Solve $\mathrm{gf}(x) = 16$.

Apply g^{-1} to both sides: $\mathrm{f}(x) = \mathrm{g}^{-1}(16)$.

$$\mathrm{g}^{-1}(16) = \frac{5(16)-4}{16+3} = \frac{80-4}{19} = \frac{76}{19} = 4.$$

So $\mathrm{f}(x) = 4$.

Case 1: On $-2 \leq x \leq 2$: $f(x) = -\frac{5}{2}(x-2) = 4 \implies x - 2 = -\frac{8}{5} \implies x = 2 - \frac{8}{5} = \frac{2}{5}$. ✓(in range)

Case 2: On $2 \leq x \leq 6$: $f(x) = x - 2 = 4 \implies x = 6$. ✓(in range)

$$x = \frac{2}{5} \quad \text{or} \quad x = 6$$

End of Worked Solutions