Question 1 (Jun 2005, Q1)

Worked Solution

$f(x) = 10 - (x+3)^2$, defined for all real x.

(i) State the range of f.

The maximum of $(x+3)^2$ is unbounded above ($\to \infty$), and its minimum is 0 (at $x = -3$). So $f(x) = 10 - (x+3)^2$ has maximum 10 and decreases without bound.

Range of f: $f(x) \leq 10$

(ii) Find the value of ff(-1).

First compute $f(-1)$:

$$f(-1) = 10 - (-1+3)^2 = 10 - (2)^2 = 10 - 4 = 6.$$

Now compute $f(6)$:

$$f(6) = 10 - (6+3)^2 = 10 - 81 = -71.$$

$ff(-1) = -71$

Question 2 (Jan 2006, Q4)

Worked Solution

$f(x) = 2 - \sqrt{x}$, $x \geq 0$.

(i) State the range of f.

Since $\sqrt{x} \geq 0$, we have $f(x) = 2 - \sqrt{x} \leq 2$. As $x \to \infty$, $f(x) \to -\infty$.

Range of f: $f(x) \leq 2$

(ii) Find the value of ff(4).

$$f(4) = 2 - \sqrt{4} = 2 - 2 = 0.$$
$$f(0) = 2 - \sqrt{0} = 2.$$

$ff(4) = 2$

(iii) Given that $|f(x)| = k$ has two distinct roots, determine the possible values of the constant k.

$|f(x)| = k$ means $|2 - \sqrt{x}| = k$, i.e. $2 - \sqrt{x} = k$ or $2 - \sqrt{x} = -k$.

Each gives $\sqrt{x} = 2 - k$ or $\sqrt{x} = 2 + k$. For a solution to exist we need the right-hand side to be ≥ 0.

$\sqrt{x} = 2 - k$ has a solution iff $k \leq 2$. $\sqrt{x} = 2 + k$ has a solution iff $k \geq -2$, which is always true for $k \geq 0$.

For two distinct roots we need both equations to contribute (each gives exactly one $x \geq 0$), and we need $2 - k \neq 2 + k$, i.e. $k \neq 0$. Also $k \geq 0$ (since $k = |f(x)|$).

The two equations give the same x only if $2 - k = 2 + k$, i.e. $k = 0$. So for $k > 0$ and $k \leq 2$ we get two distinct solutions. If $k > 2$ then $2 - k < 0$ so only one solution.

$0 < k \leq 2$

Question 3 (Jun 2009, Q5)

Worked Solution

$f(x) = 3x - 2, \quad g(x) = 3x + 7$, for all real x.

(i) Find the exact coordinates of the point at which $y = fg(x)$ meets the x-axis.

$$fg(x) = f(3x + 7) = 3(3x + 7) - 2 = 9x + 21 - 2 = 9x + 19.$$

Set $fg(x) = 0$:

$$9x + 19 = 0 \implies x = -\frac{19}{9}, \quad y = 0.$$

$$\left(-\frac{19}{9}, 0\right)$$

(ii) Find the coordinates of the point where $y = g(x)$ meets $y = g^{-1}(x)$.

The graphs of $y = g(x)$ and $y = g^{-1}(x)$ meet on the line $y = x$ (since a function and its inverse are reflections in $y = x$).

First find g^{-1}: let $y = 3x + 7 \implies x = \frac{y-7}{3}$, so $g^{-1}(x) = \frac{x-7}{3}$.

Setting $g(x) = x$: $3x + 7 = x \implies 2x = -7 \implies x = -\frac{7}{2}$, $y = -\frac{7}{2}$.

$$\left(-\frac{7}{2}, -\frac{7}{2}\right)$$

(iii) Find the exact values of x where $y = |f(x)|$ meets $y = |g(x)|$.

$|f(x)| = |3x - 2|$ and $|g(x)| = |3x + 7|$.

Set $|3x - 2| = |3x + 7|$. Squaring both sides:

$$(3x - 2)^2 = (3x + 7)^2.$$

$$9x^2 - 12x + 4 = 9x^2 + 42x + 49 \implies -12x - 42x = 49 - 4 \implies -54x = 45 \implies x = -\frac{45}{54} = -\frac{5}{6}.$$

Check: $|3(-\frac{5}{6}) - 2| = |-\frac{5}{2} - 2| = \frac{9}{2}$; $|3(-\frac{5}{6}) + 7| = |-\frac{5}{2} + 7| = \frac{9}{2}$. ✓

$y = \frac{9}{2}$.

$$x = -\frac{5}{6}, \quad y = \frac{9}{2}$$

Question 4 (Jun 2006, Q6)

Worked Solution

$f(x) = 2 - x^2$, $x \leq 0$.

(i) Evaluate $ff(-3)$.

Method 1 (numerical):

$$f(-3) = 2 - (-3)^2 = 2 - 9 = -7.$$

$$f(-7) = 2 - (-7)^2 = 2 - 49 = -47.$$

$ff(-3) = -47$

(ii) Find $f^{-1}(x)$.

Let $y = 2 - x^2$ with $x \leq 0$:

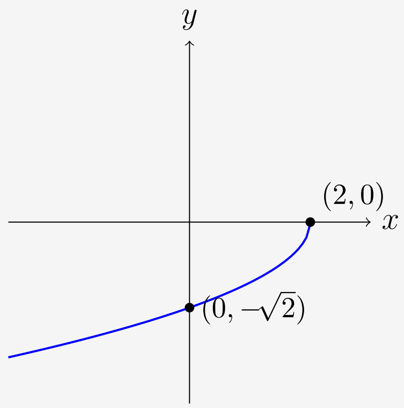
$$x^2 = 2 - y \implies x = -\sqrt{2-y} \quad \text{(taking negative root since } x \leq 0\text{)}.$$

$f^{-1}(x) = -\sqrt{2-x}$

(iii) Sketch $y = f^{-1}(x)$, indicating where the graph meets the axes.

The graph of f^{-1} is the reflection of $y = f(x)$ in $y = x$.

Key points: $f(0) = 2 \to (2, 0)$ on f^{-1}; $f^{-1}(0) = -\sqrt{2}$.



Question 5 (Jan 2011, Q9ii)

Worked Solution

$g(x) = e^{2x} + ke^{-2x}$, for all real x, where $k > 1$.

Find the range of g, giving your answer in simplified form.

To find the minimum, differentiate and set equal to zero:

$$g'(x) = 2e^{2x} - 2ke^{-2x}.$$

$$2e^{2x} - 2ke^{-2x} = 0 \implies e^{2x} = ke^{-2x} \implies e^{4x} = k \implies 4x = \ln k \implies x = \tfrac{1}{4}\ln k.$$

At this x: $e^{2x} = e^{\frac{1}{2}\ln k} = k^{1/2} = \sqrt{k}$ and $e^{-2x} = k^{-1/2} = \dfrac{1}{\sqrt{k}}$.

$$g\left(\tfrac{1}{4}\ln k\right) = \sqrt{k} + k \cdot \frac{1}{\sqrt{k}} = \sqrt{k} + \sqrt{k} = 2\sqrt{k}.$$

Since $g(x) \to \infty$ as $x \to \pm\infty$, this is a minimum.

Range of g: $g(x) \geq 2\sqrt{k}$

Question 6 (Jan 2013, Q8)

Worked Solution

$f(x) = x^2 + 4ax + a^2$, $g(x) = 4x - 2a$, for all real x, where $a > 0$.

(i) Find the range of f in terms of a.

Complete the square:

$$f(x) = (x + 2a)^2 - 4a^2 + a^2 = (x + 2a)^2 - 3a^2.$$

The minimum is $-3a^2$ at $x = -2a$.

Range of f: $f(x) \geq -3a^2$

(ii) Given that $fg(3) = 69$, find a, then find x such that $g^{-1}(x) = x$.

First find fg(3):

$$g(3) = 4(3) - 2a = 12 - 2a.$$

$$fg(3) = f(12 - 2a) = (12 - 2a)^2 + 4a(12 - 2a) + a^2.$$

Expand:

$$= 144 - 48a + 4a^2 + 48a - 8a^2 + a^2 = 144 - 3a^2.$$

Set equal to 69:

$$144 - 3a^2 = 69 \implies 3a^2 = 75 \implies a^2 = 25 \implies a = 5 \quad (a > 0).$$

Now $g(x) = 4x - 10$. Find g^{-1}: $y = 4x - 10 \implies x = \dfrac{y + 10}{4}$, so $g^{-1}(x) = \dfrac{x + 10}{4}$.

Solve $g^{-1}(x) = x$:

$$\frac{x + 10}{4} = x \implies x + 10 = 4x \implies 3x = 10 \implies x = \frac{10}{3}.$$

$a = 5$, $x = \dfrac{10}{3}$

Question 7 (Jun 2013, Q7i,ii)

Worked Solution

$f(x) = 3 + 4e^{-x}$, for all real x.

(i) State the range of f.

Since $e^{-x} > 0$ for all x, we have $4e^{-x} > 0$, so $f(x) = 3 + 4e^{-x} > 3$.

Range of f: $f(x) > 3$

(ii) Find $f^{-1}(x)$ and state its domain and range.

Let $y = 3 + 4e^{-x}$:

$$y - 3 = 4e^{-x} \implies e^{-x} = \frac{y-3}{4} \implies -x = \ln\left(\frac{y-3}{4}\right) \implies x = -\ln\left(\frac{y-3}{4}\right) = \ln\left(\frac{4}{y-3}\right).$$

$f^{-1}(x) = \ln\left(\frac{4}{x-3}\right)$

Domain of f^{-1}: $x > 3$ (equals range of f)

Range of f^{-1}: all real numbers (equals domain of f)

Question 8 (Jun 2016, Q8)

Worked Solution

$f(x) = |2x + a| + 3a, \quad g(x) = 5x - 4a$, for all real x, where $a > 0$.

(i) State the range of f and the range of g.

Since $|2x + a| \geq 0$, we have $f(x) = |2x + a| + 3a \geq 3a$.

$g(x) = 5x - 4a$ is a linear function with no restriction on x, so it takes all real values.

Range of f: $f(x) \geq 3a$. Range of g: $g(x) \in \mathbb{R}$ (all real numbers).

(ii) State why f has no inverse, and find $g^{-1}(x)$.

f is not one-to-one: for example $f(0) = a+3a = 4a$ and $f(-a) = |-a|+3a = a+3a = 4a$, so different x-values give the same output. A function must be one-to-one to have an inverse.

For g^{-1}: let $y = 5x - 4a \implies x = \dfrac{y + 4a}{5}$.

$$g^{-1}(x) = \frac{x + 4a}{5}$$

(iii) Solve $gf(x) = 31a$.

$gf(x) = g(|2x+a|+3a) = 5(|2x+a|+3a) - 4a = 5|2x+a| + 15a - 4a = 5|2x+a| + 11a$.

Set equal to $31a$:

$$5|2x + a| + 11a = 31a \implies 5|2x + a| = 20a \implies |2x + a| = 4a.$$

Case 1: $2x + a = 4a \implies 2x = 3a \implies x = \dfrac{3a}{2}$.

Case 2: $2x + a = -4a \implies 2x = -5a \implies x = -\dfrac{5a}{2}$.

$$x = \frac{3a}{2} \quad \text{or} \quad x = -\frac{5a}{2}$$

End of Worked Solutions