Functions (Domain, Range and Inverse) Exam Questions (Sheet 2) Mark Scheme

Q1.

Question Number	Scheme	Marks
(a)	$x^2-2x-3=(x-3)(x+1)$	B1
	$\mathrm{f}(x)=\dfrac{2(x-1)-(x+1)}{(x-3)(x+1)}$ $\left(or\ \dfrac{2(x-1)}{(x-3)(x+1)}-\dfrac{x+1}{(x-3)(x+1)}\right)$	M1 A1
	$=\dfrac{x-3}{(x-3)(x+1)}=\dfrac{1}{x+1}$ *	A1 cso (4)
(b)	$\left(0,\dfrac{1}{4}\right)$ Accept $0<y<\dfrac{1}{4}$, $0<\mathrm{f}(x)<\dfrac{1}{4}$ etc.	B1 B1 (2)
(c)	Let $y=\mathrm{f}(x)$ $y=\dfrac{1}{x+1}$	
	$x=\dfrac{1}{y+1}$	
	$yx+x=1$	
	$y=\dfrac{1-x}{x}$ or $\dfrac{1}{x}-1$	M1 A1
	$\mathrm{f}^{-1}(x)=\dfrac{1-x}{x}$	
	Domain of f^{-1} is $\left(0,\dfrac{1}{4}\right)$	B1 ft (3)
(d)	$\mathrm{fg}(x)=\dfrac{1}{2x^2-3+1}$	
	$\dfrac{1}{2x^2-2}=\dfrac{1}{8}$	M1
	$x^2=5$	A1
	$x=\pm\sqrt{5}$ both	A1 (3)
		(12 marks)

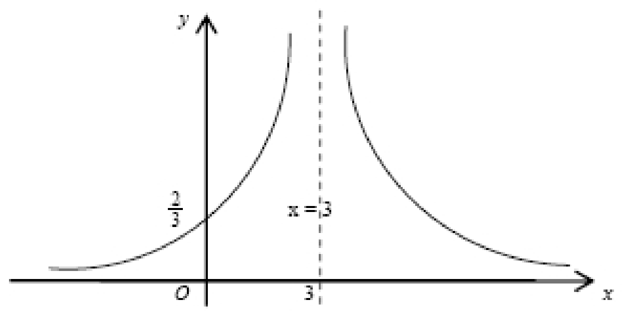
Q2.

Question No		Scheme	Marks
	(a)	$2x^2+7x-4=(2x-1)(x+4)$	B1
		$\frac{3(x+1)}{(2x-1)(x+4)}-\frac{1}{(x+4)}=\frac{3(x+1)-(2x-1)}{(2x-1)(x+4)}$	M1
		$=\frac{x+4}{(2x-1)(x+4)}$	M1
		$=\frac{1}{2x-1}$	A1*
			(4)
	(b)	$y=\frac{1}{2x-1}\Rightarrow y(2x-1)=1\Rightarrow 2xy-y=1$	
		$2xy=1+y\Rightarrow x=\frac{1+y}{2y}$	M1M1
		$y\ OR\ f^{-1}(x)=\frac{1+x}{2x}$	A1
			(3)
	(c)	x>0	B1
			(1)
	(d)	$\frac{1}{2\ln(x+1)-1}=\frac{1}{7}$	M1
		$\ln(x+1)=4$	A1
		$x=e^4-1$	M1A1
			(4)
			12 Marks

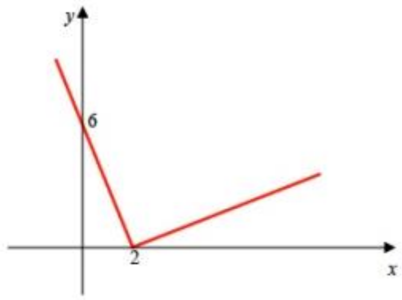
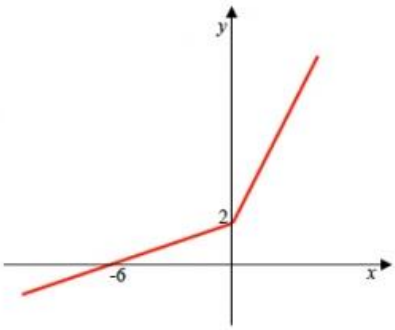
Q3.

Question Number	Scheme		Marks	
	(a) $x=1-2y^3 \Rightarrow y=\left(\frac{1-x}{2}\right)^{\frac{1}{3}}$ *or* $\sqrt[3]{\frac{1-x}{2}}$		M1 A1	(2)
	$f^{-1}:x \mapsto \left(\frac{1-x}{2}\right)^{\frac{1}{3}}$	Ignore domain		
	(b) $gf(x)=\frac{3}{1-2x^3}-4$		M1 A1	
	$=\frac{3-4(1-2x^3)}{1-2x^3}$		M1	
	$=\frac{8x^3-1}{1-2x^3}$ *	cso	A1	(4)
	$gf:x \mapsto \frac{8x^3-1}{1-2x^3}$	Ignore domain		
	(c) $8x^3-1=0$	Attempting solution of numerator = 0	M1	
	$x=\frac{1}{2}$	Correct answer and no additional answers	A1	(2)
	(d) $\frac{dy}{dx}=\frac{(1-2x^3)\times 24x^2+(8x^3-1)\times 6x^2}{(1-2x^3)^2}$		M1 A1	
	$=\frac{18x^2}{(1-2x^3)^2}$		A1	
	Solving their numerator = 0 and substituting to find y.		M1	
	$x=0,\ y=-1$		A1	(5) [13]

Q4.

Question Number	Scheme	Marks
(*a*)	Finding g(4) = k and f(k) = …. or fg(x) = $\ln\left(\frac{4}{x-3}-1\right)$	M1
	[f(2) = ln(2x2 – 1) fg(4) = ln(4 – 1)] = ln 3	A1 (2)
(*b*)	$y=\ln(2x-1) \Rightarrow e^y = 2x-1$ or $e^x = 2y-1$	M1, A1
	$f^{-1}(x) = \frac{1}{2}(e^x+1)$ Allow $y=\frac{1}{2}(e^x+1)$	A1
	Domain $x \in \Re$ [Allow $\Re$, all reals, $(-\infty, \infty)$] independent	B1 (4)
(*c*)	 Shape, and x-axis should appear to be asymptote	B1
	Equation $x = 3$ needed, may see in diagram (ignore others)	B1 ind.
	Intercept (0, $\frac{2}{3}$) no other; accept y = ⅔ (0.67) or on graph	B1 ind (3)
(*d*)	$\frac{2}{x-3}=3 \Rightarrow x = 3\frac{2}{3}$ or exact equiv.	B1
	$\frac{2}{x-3}=-3$, $\Rightarrow x = 2\frac{1}{3}$ or exact equiv.	M1, A1 (3)
	Note: $2 = 3(x+3)$ or $2 = 3(-x-3)$ o.e. is M0A0	
Alt:	Squaring to quadratic ($9x^2 - 54x + 77 = 0$) **and** solving M1; B1A1	**(12 marks)**

Q5.

Question Number	Scheme		Marks
(a)	$y = \dfrac{3-2x}{x-5} \Rightarrow y(x-5) = 3-2x$	Attempt to make x (or swapped y) the subject	M1
	$xy - 5y = 3 - 2x$		
	$\Rightarrow xy + 2x = 3 + 5y \Rightarrow x(y+2) = 3+5y$	Collect x terms together and factorise.	M1
	$\Rightarrow x = \dfrac{3+5y}{y+2} \quad \therefore \mathrm{f}^{-1}(x) = \underline{\dfrac{3+5x}{x+2}}$	$\underline{\dfrac{3+5x}{x+2}}$	A1 oe (3)
(b)	Range of g is $-9 \le \mathrm{g}(x) \le 4$ or $-9 \le y \le 4$	Correct Range	B1 (1)
(c)		Deduces that $\mathrm{g}(2)$ is 0. Seen or implied.	M1
	$\mathrm{g\,g}(2) = \mathrm{g}(0) = -6$, from sketch.	-6	A1 (2)
(d)	$\mathrm{fg}(8) = \mathrm{f}(4)$	Correct order g followed by f	M1
	$= \dfrac{3-4(2)}{4-5} = \dfrac{-5}{-1} = \underline{5}$	5	A1 (2)
(e)(i)		Correct shape 	B1
		$(2, \{0\}), (\{0\}, 6)$	B1
Question Number	Scheme		Marks
(e)(ii)		Correct shape 	B1
		Graph goes through $(\{0\}, 2)$ and $(-6, \{0\})$ which are marked.	B1 (4)
(f)	Domain of g^{-1} is $-9 \le x \le 4$	Either correct answer or a follow through from part (b) answer	B1$\sqrt{}$ (1) [13]

Q6.

Question Number	Scheme	Marks
(a)	$f(x) \geqslant 3$	M1A1
		(2)
(b)	An attempt to find $2\|3-4x\|+3$ when $x=1$	M1
	Correct answer $fg(1)=5$	A1
		(2)
(c)	$y=3-4x \Rightarrow 4x=3-y \Rightarrow x=\dfrac{3-y}{4}$	M1
	$g^{-1}(x)=\dfrac{3-x}{4}$	A1
		(2)
(d)	$[g(x)]^2=(3-4x)^2$	B1
	$gg(x)=3-4(3-4x)$	M1
	$gg(x)+[g(x)]^2=0 \Rightarrow -9+16x+9-24x+16x^2=0$	
	$16x^2-8x=0$	A1
	$8x(2x-1)=0 \Rightarrow x=0, 0.5$ oe	M1A1
		(5)
		(11 marks)

Q7.

Question Number	Scheme	Marks
(a)	$y..3$	B1
		(1)
(b)	$y=3+\sqrt{x+2}\Rightarrow y-3=\sqrt{x+2}\Rightarrow x=(y-3)^2-2$	M1 A1
	$\Rightarrow g^{-1}(x)=(x-3)^2-2, \quad \text{with } x..3$	A1
		(3)
(c)	$g(x)=x\Rightarrow 3+\sqrt{x+2}=x$	
	$\Rightarrow x+2=(x-3)^2\Rightarrow x^2-7x+7=0$	M1, A1
	$\Rightarrow x=\frac{7\pm\sqrt{21}}{2}\Rightarrow x=\frac{7+\sqrt{21}}{2}$ only	M1, A1
		(4)
(d)	$a=\frac{7+\sqrt{21}}{2}$	B1 ft
		(1)
		9 marks
(c) Alt	Solves $g^{-1}(x)=x\Rightarrow(x-3)^2-2=x$	
	$\Rightarrow x^2-7x+7=0$	M1, A1
	$\Rightarrow x=\frac{7\pm\sqrt{21}}{2}\Rightarrow x=\frac{7+\sqrt{21}}{2}$ only	dM1, A1
		(4)

Q8.

Question Number	Scheme	Marks
(a)	$f(x) > k^2$	B1 (1)
(b)	$y = e^{2x} + k^2 \Rightarrow e^{2x} = y - k^2$	M1
	$\Rightarrow x = \frac{1}{2}\ln(y - k^2)$	dM1
	$\Rightarrow f^{-1}(x) = \frac{1}{2}\ln(x - k^2), \quad x > k^2$	A1 (3)
(c)	$\ln 2x + \ln 2x^2 + \ln 2x^3 = 6$	M1
	$\Rightarrow \ln 8x^6 = 6$	M1
	$\Rightarrow 8x^6 = e^6 \Rightarrow x = ..$	M1
	$\Rightarrow x = \left(\frac{e}{\sqrt[6]{8}}\right) = \frac{e}{\sqrt{2}}$ (Ignore any reference to $-\frac{e}{\sqrt{2}}$)	A1 (4)
(d)	$fg(x) = e^{2\times\ln(2x)} + k^2$	M1
	$\Rightarrow fg(x) = (2x)^2 + k^2 = 4x^2 + k^2$	A1 (2)
(e)	$fg(x) = 2k^2 \Rightarrow 4x^2 + k^2 = 2k^2$	
	$\Rightarrow 4x^2 = k^2 \Rightarrow x = ..$	M1
	$\Rightarrow x = \frac{k}{2}$ **only**	A1 (2) **12 marks**
(alt c)	$\ln 2x + \ln 2x^2 + \ln 2x^3 = 6$	M1
	$\Rightarrow \ln 2 + \ln x + \ln 2 + 2\ln x + \ln 2 + 3\ln x = 6$	
	$\Rightarrow 3\ln 2 + 6\ln x = 6$	
	$\Rightarrow \ln x = 1 - \frac{1}{2}\ln 2$	M1
	$\Rightarrow x = e^{1-\frac{1}{2}\ln 2}, = \frac{e}{\sqrt{2}}$ (Ignore any reference to $-\frac{e}{\sqrt{2}}$)	M1, A1 (4)
(alt e)	$\Rightarrow 2\ln(2x) = \ln(2k^2 - k^2)$	
	$\Rightarrow \ln(2x)^2 = \ln(k^2), \Rightarrow 4x^2 = k^2 \Rightarrow x = \frac{k}{2}$	M1, A1

Q9.

Question Number	Scheme	Marks
(a)	$0 \leqslant f(x) \leqslant 10$	B1
		(1)
(b)	$ff(0) = f(5), = 3$	B1,B1
		(2)
(c)	$y = \dfrac{4+3x}{5-x} \Rightarrow y(5-x) = 4+3x$	
	$\Rightarrow 5y - 4 = xy + 3x$	M1
	$\Rightarrow 5y-4 = x(y+3) \Rightarrow x = \dfrac{5y-4}{y+3}$	dM1
	$g^{-1}(x) = \dfrac{5x-4}{3+x}$	A1
		(3)
(d)	$gf(x) = 16 \Rightarrow f(x) = g^{-1}(16) = 4$ oe	M1A1
	$f(x) = 4 \Rightarrow x = 6$	B1
	$f(x) = 4 \Rightarrow 5 - 2.5x = 4 \Rightarrow x = 0.4$ oe	M1A1
		(5)
		(11 marks)
Alt 1 to (d)	$gf(x) = 16 \Rightarrow \dfrac{4+3(ax+b)}{5-(ax+b)} = 16$	M1
	$ax + b = x - 2$ *or* $5 - 2.5x$	A1
	$\Rightarrow x = 6$	B1
	$\dfrac{4+3(5-2.5x)}{5-(5-2.5x)} = 16 \Rightarrow x = \dots$	M1
	$\Rightarrow x = 0.4$ oe	A1 (5)