

Question 1 (Jan 2006, Q3)**Worked Solution**

(a) Differentiate $x^2(x+1)^6$ with respect to x .

Product rule: $u = x^2$, $v = (x+1)^6$:

$$\frac{dy}{dx} = 2x(x+1)^6 + x^2 \cdot 6(x+1)^5 = 2x(x+1)^6 + 6x^2(x+1)^5$$

$$\frac{dy}{dx} = 2x(x+1)^6 + 6x^2(x+1)^5$$

(b) Find the gradient of $y = \frac{x^2+3}{x^2-3}$ at $x = 1$.

Quotient rule: $u = x^2 + 3$, $v = x^2 - 3$:

$$\frac{dy}{dx} = \frac{2x(x^2-3) - (x^2+3) \cdot 2x}{(x^2-3)^2} = \frac{2x[(x^2-3) - (x^2+3)]}{(x^2-3)^2} = \frac{2x(-6)}{(x^2-3)^2} = \frac{-12x}{(x^2-3)^2}$$

$$\text{At } x = 1: \frac{dy}{dx} = \frac{-12}{(1-3)^2} = \frac{-12}{4} = -3.$$

$$\text{Gradient} = -3$$

Question 2 (Jun 2006, Q1)**Worked Solution**

Find the equation of the tangent to $y = \sqrt{4x + 1}$ at $(2, 3)$.

Write $y = (4x + 1)^{1/2}$. Chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(4x + 1)^{-1/2} \cdot 4 = \frac{2}{(4x + 1)^{1/2}}$$

At $x = 2$: $\frac{dy}{dx} = \frac{2}{\sqrt{9}} = \frac{2}{3}$.

Tangent through $(2, 3)$: $y - 3 = \frac{2}{3}(x - 2) \implies y = \frac{2}{3}x + \frac{5}{3}$.

$$y = \frac{2}{3}x + \frac{5}{3} \quad (\text{or } 2x - 3y + 5 = 0)$$

Question 3 (Jan 2007, Q1)**Worked Solution**

Find the equation of the tangent to $y = \frac{2x+1}{3x-1}$ at $(1, \frac{3}{2})$, in the form $ax + by + c = 0$.

Quotient rule: $u = 2x + 1$, $v = 3x - 1$:

$$\frac{dy}{dx} = \frac{2(3x-1) - 3(2x+1)}{(3x-1)^2} = \frac{6x-2-6x-3}{(3x-1)^2} = \frac{-5}{(3x-1)^2}$$

At $x = 1$: gradient = $\frac{-5}{4}$.

Tangent: $y - \frac{3}{2} = -\frac{5}{4}(x - 1) \implies 4y - 6 = -5x + 5 \implies 5x + 4y - 11 = 0$.

$$5x + 4y - 11 = 0$$

Question 4 (Jun 2007, Q1)**Worked Solution**

(i) Differentiate $x^3(x+1)^5$.

Product rule: $u = x^3$, $v = (x+1)^5$:

$$\frac{dy}{dx} = 3x^2(x+1)^5 + 5x^3(x+1)^4$$

$$3x^2(x+1)^5 + 5x^3(x+1)^4$$

(ii) Differentiate $\sqrt{3x^4+1}$.

Write $y = (3x^4+1)^{1/2}$. Chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3x^4+1)^{-1/2} \cdot 12x^3 = \frac{6x^3}{(3x^4+1)^{1/2}} = 6x^3(3x^4+1)^{-1/2}$$

$$\frac{6x^3}{\sqrt{3x^4+1}}$$

Question 5 (Jun 2008, Q3)

Worked Solution

Find the equation of the tangent to $y = x^2 \ln x$ at the point with x -coordinate e .

Product rule: $u = x^2$, $v = \ln x$:

$$\frac{dy}{dx} = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$$

At $x = e$: $\frac{dy}{dx} = 2e \ln e + e = 2e + e = 3e$.

y -coordinate: $y = e^2 \ln e = e^2$.

Tangent: $y - e^2 = 3e(x - e) \implies y = 3ex - 3e^2 + e^2 = 3ex - 2e^2$.

$$y = 3ex - 2e^2$$

Question 6 (Jan 2010, Q5)

Worked Solution

$$y = (x^2 + 1)^8.$$

(i) Find $\frac{dy}{dx}$ and show the only stationary point is at $x = 0$.

Chain rule:

$$\frac{dy}{dx} = 8(x^2 + 1)^7 \cdot 2x = 16x(x^2 + 1)^7$$

Set to zero: $16x(x^2 + 1)^7 = 0$. Since $(x^2 + 1)^7 \geq 1 > 0$, we need $x = 0$. So the only stationary point is at $x = 0$. \square

(ii) Find $\frac{d^2y}{dx^2}$ and its value at the stationary point.

Product rule on $\frac{dy}{dx} = 16x(x^2 + 1)^7$:

$$\frac{d^2y}{dx^2} = 16(x^2 + 1)^7 + 16x \cdot 7(x^2 + 1)^6 \cdot 2x = 16(x^2 + 1)^7 + 224x^2(x^2 + 1)^6$$

At $x = 0$:

$$\frac{d^2y}{dx^2} = 16(1)^7 + 0 = 16$$

$$\frac{d^2y}{dx^2} = 16(x^2 + 1)^7 + 224x^2(x^2 + 1)^6; \text{ value at stationary point} = 16$$

Question 7 (Jun 2010, Q1)**Worked Solution**

Find $\frac{dy}{dx}$ in each case:

(i) $y = x^3 e^{2x}$

Product rule:

$$\frac{dy}{dx} = 3x^2 e^{2x} + 2x^3 e^{2x} = x^2 e^{2x} (3 + 2x)$$

$$\frac{dy}{dx} = 3x^2 e^{2x} + 2x^3 e^{2x}$$

(ii) $y = \ln(3 + 2x^2)$

Chain rule:

$$\frac{dy}{dx} = \frac{4x}{3 + 2x^2}$$

$$\frac{dy}{dx} = \frac{4x}{3 + 2x^2}$$

(iii) $y = \frac{x}{2x + 1}$

Quotient rule:

$$\frac{dy}{dx} = \frac{(2x + 1) - x \cdot 2}{(2x + 1)^2} = \frac{1}{(2x + 1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(2x + 1)^2}$$

Question 8 (Jan 2013, Q1)**Worked Solution**

Find the gradient at $x = 2$ for each curve.

(i) $y = \frac{3x}{2x + 1}$

$$\frac{dy}{dx} = \frac{3(2x + 1) - 3x \cdot 2}{(2x + 1)^2} = \frac{3}{(2x + 1)^2}$$

At $x = 2$: $\frac{3}{25}$.

Gradient = $\frac{3}{25}$

(ii) $y = \sqrt{4x^2 + 9}$

$$\frac{dy}{dx} = \frac{8x}{2\sqrt{4x^2 + 9}} = \frac{4x}{\sqrt{4x^2 + 9}}$$

At $x = 2$: $\frac{8}{\sqrt{25}} = \frac{8}{5}$.

Gradient = $\frac{8}{5}$

Question 9 (Jan 2013, Q4)**Worked Solution**

$$m = 250e^{0.021t}$$

(i) Find the time for the mass to double, and deduce the time to increase 8-fold.

$$\text{Double: } 250e^{0.021t} = 500 \implies e^{0.021t} = 2 \implies t = \frac{\ln 2}{0.021} \approx 33 \text{ hours.}$$

8-fold = doubling three times, so $t = 3 \times 33 = 99$ hours.

Time to double ≈ 33 hours; time to increase 8-fold ≈ 99 hours

(ii) Find the rate of increase when $m = 400$ grams.

$$\frac{dm}{dt} = 250 \times 0.021 e^{0.021t} = 0.021m.$$

When $m = 400$: rate = $0.021 \times 400 = 8.4$ g/hour.

Rate = 8.4 g per hour

Question 10 (Jun 2014, Q1)**Worked Solution**

$y = 4x^2 \ln x$. Find $\frac{d^2y}{dx^2}$ when $x = e^2$.

First derivative (product rule):

$$\frac{dy}{dx} = 8x \ln x + 4x^2 \cdot \frac{1}{x} = 8x \ln x + 4x$$

Second derivative (product rule on $8x \ln x$):

$$\frac{d^2y}{dx^2} = 8 \ln x + 8x \cdot \frac{1}{x} + 4 = 8 \ln x + 8 + 4 = 8 \ln x + 12$$

At $x = e^2$: $\frac{d^2y}{dx^2} = 8 \times 2 + 12 = 16 + 12 = 28$.

$$\frac{d^2y}{dx^2} = 28$$

Question 11 (Jun 2015, Q1)**Worked Solution**

Find the equation of the tangent to $y = \frac{5x + 4}{3x - 8}$ at $(2, -7)$.

Quotient rule:

$$\frac{dy}{dx} = \frac{5(3x - 8) - 3(5x + 4)}{(3x - 8)^2} = \frac{15x - 40 - 15x - 12}{(3x - 8)^2} = \frac{-52}{(3x - 8)^2}$$

At $x = 2$: gradient = $\frac{-52}{(6 - 8)^2} = \frac{-52}{4} = -13$.

Tangent: $y + 7 = -13(x - 2) \implies y = -13x + 26 - 7 = -13x + 19$.

$$y = -13x + 19 \quad (\text{or } 13x + y - 19 = 0)$$

Question 12 (Jun 2016, Q1)**Worked Solution**

Find the equation of the tangent to $y = 3x^2(x+2)^6$ at $(-1, 3)$, in the form $y = mx + c$.

Product rule: $u = 3x^2$, $v = (x+2)^6$:

$$\frac{dy}{dx} = 6x(x+2)^6 + 3x^2 \cdot 6(x+2)^5 = 6x(x+2)^6 + 18x^2(x+2)^5$$

At $x = -1$: $(x+2) = 1$, so:

$$\frac{dy}{dx} = 6(-1)(1)^6 + 18(1)(1)^5 = -6 + 18 = 12$$

Tangent: $y - 3 = 12(x + 1) \implies y = 12x + 15$.

$$y = 12x + 15$$

Question 13 (OCR 4753, Jan 2009, Q2i)

Worked Solution

Differentiate $x \cos 2x$ with respect to x .

Product rule: $u = x$, $v = \cos 2x$:

$$\frac{dy}{dx} = \cos 2x + x \cdot (-2 \sin 2x) = \cos 2x - 2x \sin 2x$$

$$\frac{dy}{dx} = \cos 2x - 2x \sin 2x$$

Question 14 (OCR 4753, Jan 2012, Q1)

Worked Solution

Differentiate $x^2 \tan 2x$.

Product rule: $u = x^2$, $v = \tan 2x$ (so $v' = 2 \sec^2 2x$):

$$\frac{dy}{dx} = 2x \tan 2x + 2x^2 \sec^2 2x$$

$$\frac{dy}{dx} = 2x \tan 2x + 2x^2 \sec^2 2x$$

Question 15 (OCR 4753, Jun 2010, Q3)

Worked Solution

(i) Differentiate $\sqrt{1 + 3x^2}$.

Write $y = (1 + 3x^2)^{1/2}$. Chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(1 + 3x^2)^{-1/2} \cdot 6x = \frac{3x}{\sqrt{1 + 3x^2}}$$

$$\frac{dy}{dx} = \frac{3x}{\sqrt{1 + 3x^2}}$$

(ii) Show that the derivative of $x\sqrt{1 + 3x^2}$ is $\frac{1 + 6x^2}{\sqrt{1 + 3x^2}}$.

Product rule: $u = x$, $v = (1 + 3x^2)^{1/2}$, using result from (i):

$$\frac{dy}{dx} = (1 + 3x^2)^{1/2} + x \cdot \frac{3x}{\sqrt{1 + 3x^2}} = \frac{(1 + 3x^2) + 3x^2}{\sqrt{1 + 3x^2}} = \frac{1 + 6x^2}{\sqrt{1 + 3x^2}}$$

□

End of Worked Solutions