Question 1

Worked Solution

$y = \dfrac{5x^2 + 10x}{(x+1)^2}$, $x \neq -1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$, find A and n.

Quotient rule: $u = 5x^2 + 10x = 5x(x+2)$, $v = (x+1)^2$, $u' = 10x + 10$, $v' = 2(x+1)$:

$$\frac{dy}{dx} = \frac{(x+1)^2(10x+10) - (5x^2+10x)\cdot 2(x+1)}{(x+1)^4}$$

Factor out $(x+1)$ from the numerator:

$$= \frac{(x+1)\big[(x+1)(10x+10) - 2(5x^2+10x)\big]}{(x+1)^4} = \frac{(x+1)(10x+10) - 2(5x^2+10x)}{(x+1)^3}$$

Expand the numerator:

$$(x+1)(10x+10) - 2(5x^2+10x) = 10x^2 + 20x + 10 - 10x^2 - 20x = 10$$

Therefore:

$$\frac{dy}{dx} = \frac{10}{(x+1)^3}$$

$A = 10$, $n = 3$

(b) Deduce the range of values of x for which $\dfrac{dy}{dx} < 0$.

Since $A = 10 > 0$, $\dfrac{dy}{dx} = \dfrac{10}{(x+1)^3} < 0$ when $(x+1)^3 < 0$, i.e. $x + 1 < 0$.

$x < -1$

Question 2

Worked Solution

Given $y = x(2x+1)^4$, show that $\dfrac{dy}{dx} = (2x+1)^n(Ax+B)$ and find n, A, B.

Product rule: $u = x$, $v = (2x+1)^4$:

$$\frac{dy}{dx} = (2x+1)^4 + x \cdot 4(2x+1)^3 \cdot 2 = (2x+1)^4 + 8x(2x+1)^3$$

Factor out $(2x+1)^3$:

$$= (2x+1)^3\big[(2x+1) + 8x\big] = (2x+1)^3(10x+1)$$

$n = 3,\ A = 10,\ B = 1$

Question 3

Worked Solution

$y = (2x-3)^5$, and the point P has coordinates $(w, -32)$.

(a) Find w.

$(2w-3)^5 = -32 = (-2)^5 \implies 2w - 3 = -2 \implies w = \frac{1}{2}$.

$$w = \frac{1}{2}$$

(b) Find the equation of the tangent at P.

Chain rule: $\dfrac{dy}{dx} = 5(2x-3)^4 \cdot 2 = 10(2x-3)^4$.

At $x = \frac{1}{2}$: $(2 \cdot \frac{1}{2} - 3)^4 = (-2)^4 = 16$, so gradient $= 10 \times 16 = 160$.

Tangent: $y - (-32) = 160\left(x - \frac{1}{2}\right) \implies y = 160x - 80 - 32 = 160x - 112$.

$$y = 160x - 112$$

Question 4

Worked Solution

Differentiate with respect to x:

(a) $\ln(x^2 + 3x + 5)$

Chain rule:

$$\frac{dy}{dx} = \frac{2x + 3}{x^2 + 3x + 5}$$

$$\frac{2x + 3}{x^2 + 3x + 5}$$

(b) $\dfrac{\cos x}{x^2}$

Quotient rule: $u = \cos x$, $v = x^2$:

$$\frac{dy}{dx} = \frac{-\sin x \cdot x^2 - \cos x \cdot 2x}{x^4} = \frac{-x^2 \sin x - 2x \cos x}{x^4} = \frac{-x \sin x - 2 \cos x}{x^3}$$

$$\frac{-x \sin x - 2 \cos x}{x^3}$$

Question 5

Worked Solution

Differentiate with respect to x, simplest form:

(a) $x^2 \ln(3x)$

Product rule: $u = x^2$, $v = \ln(3x)$, $v' = \dfrac{1}{x}$:

$$\frac{dy}{dx} = 2x\ln(3x) + x^2 \cdot \frac{1}{x} = 2x\ln(3x) + x = x(2\ln(3x) + 1)$$

$x(2\ln(3x) + 1)$

(b) $\dfrac{\sin 4x}{x^3}$

Quotient rule: $u = \sin 4x$, $v = x^3$:

$$\frac{dy}{dx} = \frac{4x^3\cos 4x - 3x^2\sin 4x}{x^6} = \frac{4x\cos 4x - 3\sin 4x}{x^4}$$

$\dfrac{4x\cos 4x - 3\sin 4x}{x^4}$

Question 6

Worked Solution

$y = e^{2x} \tan x$, $x \neq (2n+1)\dfrac{\pi}{2}$.

(a) Show that turning points occur where $\tan x = -1$.

Product rule: $u = e^{2x}$, $v = \tan x$:

$$\frac{dy}{dx} = 2e^{2x} \tan x + e^{2x} \sec^2 x = e^{2x}(2\tan x + \sec^2 x)$$

Set to zero. Since $e^{2x} > 0$:

$$2\tan x + \sec^2 x = 0$$

Use $\sec^2 x = 1 + \tan^2 x$:

$$2\tan x + 1 + \tan^2 x = 0 \implies (\tan x + 1)^2 = 0 \implies \tan x = -1 \qquad \square$$

(b) Find the equation of the tangent at $x = 0$.

At $x = 0$: $y = e^0 \tan 0 = 0$.

$\dfrac{dy}{dx}\Big|_{x=0} = e^0(2\tan 0 + \sec^2 0) = 1 \cdot (0 + 1) = 1$.

Tangent: $y = x$.

$y = x$

Question 7

Worked Solution

(i)(a) Differentiate $x^2 \cos 3x$.

Product rule: $u = x^2$, $v = \cos 3x$:

$$\frac{dy}{dx} = 2x \cos 3x - 3x^2 \sin 3x$$

$2x \cos 3x - 3x^2 \sin 3x$

(i)(b) Differentiate $\dfrac{\ln(x^2+1)}{x^2+1}$.

Quotient rule: $u = \ln(x^2+1)$, $u' = \dfrac{2x}{x^2+1}$; $v = x^2+1$, $v' = 2x$:

$$\frac{dy}{dx} = \frac{\frac{2x}{x^2+1}(x^2+1) - \ln(x^2+1) \cdot 2x}{(x^2+1)^2} = \frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2} = \frac{2x(1-\ln(x^2+1))}{(x^2+1)^2}$$

$\dfrac{2x\big(1-\ln(x^2+1)\big)}{(x^2+1)^2}$

(ii) $y = \sqrt{4x+1}$, $x > -\frac{1}{4}$. Find the equation of the tangent at P $(x = 2)$ in the form $ax + by + c = 0$.

At P: $y = \sqrt{9} = 3$.

$\dfrac{dy}{dx} = \dfrac{1}{2}(4x+1)^{-1/2} \cdot 4 = \dfrac{2}{\sqrt{4x+1}}$.

At $x = 2$: gradient $= \dfrac{2}{3}$.

Tangent: $y - 3 = \dfrac{2}{3}(x-2) \implies 3y - 9 = 2x - 4 \implies 2x - 3y + 5 = 0 \cdot$.

$2x - 3y + 5 = 0$

Question 8

Worked Solution

$y = \dfrac{3}{(5-3x)^2}$, $x \neq \frac{5}{3}$. Find the equation of the normal at P $(x = 2)$ in the form $ax + by + c = 0$.

Write $y = 3(5-3x)^{-2}$. Chain rule:

$$\frac{dy}{dx} = 3 \cdot (-2)(5-3x)^{-3} \cdot (-3) = \frac{18}{(5-3x)^3}$$

At $x = 2$: $(5-6)^3 = (-1)^3 = -1$, so gradient of tangent $= \dfrac{18}{-1} = -18$.

y-coordinate: $y = \dfrac{3}{(-1)^2} = 3$.

Gradient of normal $= \dfrac{1}{18}$.

Normal: $y - 3 = \dfrac{1}{18}(x-2) \implies 18y - 54 = x - 2 \implies x - 18y + 52 = 0 \cdot$.

$x - 18y + 52 = 0$

Question 9

Worked Solution

$y = \dfrac{x-4}{2+\sqrt{x}}$, $x > 0$. Show that $\dfrac{dy}{dx} = \dfrac{1}{A\sqrt{x}}$ and find A.

Quotient rule: $u = x - 4$, $v = 2 + \sqrt{x} = 2 + x^{1/2}$, $u' = 1$, $v' = \dfrac{1}{2\sqrt{x}}$:

$$\frac{dy}{dx} = \frac{(2+\sqrt{x}) - (x-4)\cdot\dfrac{1}{2\sqrt{x}}}{(2+\sqrt{x})^2}$$

Multiply numerator and denominator by $2\sqrt{x}$:

$$= \frac{2\sqrt{x}(2+\sqrt{x}) - (x-4)}{2\sqrt{x}(2+\sqrt{x})^2}$$

Numerator: $4\sqrt{x} + 2x - x + 4 = x + 4\sqrt{x} + 4 = (\sqrt{x}+2)^2 = (2+\sqrt{x})^2$.

$$\frac{dy}{dx} = \frac{(2+\sqrt{x})^2}{2\sqrt{x}(2+\sqrt{x})^2} = \frac{1}{2\sqrt{x}}$$

$A = 2$

Question 10

Worked Solution

$y = e^{\sqrt{3}x} \sin 3x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$.

(a) Find the x-coordinate of the turning point P for $x > 0$, as a multiple of π.

Product rule: $u = e^{\sqrt{3}x}$, $v = \sin 3x$:

$$\frac{dy}{dx} = \sqrt{3}\, e^{\sqrt{3}x} \sin 3x + 3e^{\sqrt{3}x} \cos 3x = e^{\sqrt{3}x}(\sqrt{3} \sin 3x + 3 \cos 3x)$$

Set to zero (since $e^{\sqrt{3}x} > 0$):

$$\sqrt{3} \sin 3x + 3 \cos 3x = 0 \implies \tan 3x = -\sqrt{3}$$

$$3x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \implies x = \frac{2\pi}{9}$$

$x = \frac{2\pi}{9}$

(b) Find the equation of the normal to C at $x = 0$.

At $x = 0$: $y = e^0 \sin 0 = 0$.

$\frac{dy}{dx}\Big|_{x=0} = e^0(\sqrt{3} \cdot 0 + 3 \cdot 1) = 3$.

Gradient of normal $= -\frac{1}{3}$.

Normal: $y = -\frac{1}{3}x$.

$y = -\frac{1}{3}x$

End of Worked Solutions