

Question 1 (Jun 2005, Q6)**Worked Solution**

(a) Find the exact value of the x -coordinate of the stationary point of the curve $y = x \ln x$.

Use the product rule with $u = x$, $v = \ln x$:

$$\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

Set $\frac{dy}{dx} = 0$:

$$\ln x + 1 = 0 \implies \ln x = -1 \implies x = e^{-1}$$

$$x = e^{-1}$$

(b) Show that the curve $y = \frac{4x + c}{4x - c}$ has no stationary points ($c \neq 0$).

Use the quotient rule with $u = 4x + c$, $v = 4x - c$:

$$\frac{dy}{dx} = \frac{(4x - c) \cdot 4 - (4x + c) \cdot 4}{(4x - c)^2} = \frac{16x - 4c - 16x - 4c}{(4x - c)^2} = \frac{-8c}{(4x - c)^2}$$

Since $c \neq 0$, the numerator $-8c \neq 0$, and the denominator $(4x - c)^2 > 0$ wherever it is defined. Therefore $\frac{dy}{dx} \neq 0$ for all valid x , so the curve has no stationary points. \square

Question 2 (Jun 2007, Q8i,ii)

Worked Solution

(i) Given $y = \frac{4 \ln x - 3}{4 \ln x + 3}$, show that $\frac{dy}{dx} = \frac{24}{x(4 \ln x + 3)^2}$.

Use the quotient rule with $u = 4 \ln x - 3$, $v = 4 \ln x + 3$, so $u' = \frac{4}{x}$, $v' = \frac{4}{x}$:

$$\frac{dy}{dx} = \frac{(4 \ln x + 3) \cdot \frac{4}{x} - (4 \ln x - 3) \cdot \frac{4}{x}}{(4 \ln x + 3)^2} = \frac{\frac{4}{x} [(4 \ln x + 3) - (4 \ln x - 3)]}{(4 \ln x + 3)^2} = \frac{\frac{4}{x} \cdot 6}{(4 \ln x + 3)^2} = \frac{24}{x(4 \ln x + 3)^2}$$

□

(ii) Find the exact value of the gradient of the curve at the point where it crosses the x -axis.

The curve crosses the x -axis when $y = 0$, i.e. $4 \ln x - 3 = 0$, so $\ln x = \frac{3}{4}$, giving $x = e^{3/4}$.

Substitute into the derivative:

$$\frac{dy}{dx} = \frac{24}{e^{3/4} \cdot (4 \cdot \frac{3}{4} + 3)^2} = \frac{24}{e^{3/4} \cdot 6^2} = \frac{24}{36 e^{3/4}} = \frac{2}{3} e^{-3/4}$$

Gradient = $\frac{2}{3} e^{-3/4}$

Question 3 (Jan 2008, Q7)

Worked Solution

The curve has equation $y = \frac{xe^{2x}}{x+k}$, $k \neq 0$.

(i) Differentiate xe^{2x} and show that $\frac{dy}{dx} = \frac{e^{2x}(2x^2 + 2kx + k)}{(x+k)^2}$.

First, differentiate xe^{2x} using the product rule:

$$\frac{d}{dx}(xe^{2x}) = e^{2x} + 2xe^{2x} = e^{2x}(1 + 2x)$$

Now apply the quotient rule to $y = \frac{xe^{2x}}{x+k}$ with $u = xe^{2x}$, $v = x+k$:

$$\frac{dy}{dx} = \frac{(x+k) \cdot e^{2x}(1+2x) - xe^{2x} \cdot 1}{(x+k)^2} = \frac{e^{2x}[(x+k)(1+2x) - x]}{(x+k)^2}$$

Expand the numerator bracket:

$$(x+k)(1+2x) - x = x + 2x^2 + k + 2kx - x = 2x^2 + 2kx + k$$

Therefore:

$$\frac{dy}{dx} = \frac{e^{2x}(2x^2 + 2kx + k)}{(x+k)^2} \quad \square$$

(ii) Given that the curve has exactly one stationary point, find k and the exact coordinates.

At a stationary point, $\frac{dy}{dx} = 0$, so $2x^2 + 2kx + k = 0$ (since $e^{2x} > 0$).

For exactly one stationary point, the discriminant equals zero:

$$\Delta = (2k)^2 - 4 \cdot 2 \cdot k = 4k^2 - 8k = 4k(k-2) = 0$$

Since $k \neq 0$, we get $k = 2$.

With $k = 2$: $2x^2 + 4x + 2 = 0 \implies (x+1)^2 = 0 \implies x = -1$.

y -coordinate: $y = \frac{(-1)e^{-2}}{-1+2} = -e^{-2}$.

$k = 2$; stationary point at $(-1, -e^{-2})$

Question 4 (Jan 2013, Q7)

Worked Solution

The curve has equation $x = (y + 4) \ln(2y + 3)$.

(i) Find $\frac{dx}{dy}$.

Use the product rule with $u = y + 4$, $v = \ln(2y + 3)$:

$$\frac{dx}{dy} = \ln(2y + 3) + (y + 4) \cdot \frac{2}{2y + 3}$$

$$\frac{dx}{dy} = \ln(2y + 3) + \frac{2(y + 4)}{2y + 3}$$

(ii) Find the gradient of the curve at A and B (2 d.p.).

The gradient of the curve is $\frac{dy}{dx} = \frac{1}{dx/dy}$.

Point A (on the x -axis): $y = 0$.

$$\left. \frac{dx}{dy} \right|_{y=0} = \ln 3 + \frac{2 \cdot 4}{3} = \ln 3 + \frac{8}{3}$$

$$\left. \frac{dy}{dx} \right|_A = \frac{1}{\ln 3 + 8/3} \approx \frac{1}{1.0986 + 2.6667} \approx \frac{1}{3.7653} \approx 0.27$$

Point B (on the y -axis): $x = 0$, so $(y + 4) \ln(2y + 3) = 0$.

Either $y = -4$ or $\ln(2y + 3) = 0 \Rightarrow y = -1$. Since $y = -4$ gives $\ln(-5)$ undefined, $y = -1$.

$$\left. \frac{dx}{dy} \right|_{y=-1} = \ln 1 + \frac{2 \cdot 3}{1} = 0 + 6 = 6$$

$$\left. \frac{dy}{dx} \right|_B = \frac{1}{6} \approx 0.17$$

Gradient at $A \approx 0.27$; gradient at $B \approx 0.17$

Question 5 (Jan 2011, Q9)

Worked Solution

$$f(x) = e^{2x} - 3e^{-2x}$$

(i)(a) Show that $f'(x) > 0$ for all x .

$$f'(x) = 2e^{2x} + 6e^{-2x}$$

Since $e^{2x} > 0$ and $e^{-2x} > 0$ for all x , both terms are positive, so $f'(x) > 0$ for all x . \square

(i)(b) Show that the set of values of x for which $f''(x) > 0$ is the same as the set for which $f(x) > 0$.

$$f''(x) = 4e^{2x} - 12e^{-2x}$$

$$f''(x) > 0 \iff 4e^{2x} > 12e^{-2x} \iff e^{4x} > 3 \iff x > \frac{1}{4} \ln 3.$$

$$f(x) > 0 \iff e^{2x} > 3e^{-2x} \iff e^{4x} > 3 \iff x > \frac{1}{4} \ln 3.$$

Both conditions give the same set $x > \frac{1}{4} \ln 3$. \square

(ii) $g(x) = e^{2x} + ke^{-2x}$, $k > 1$. Find the range of g .

Find the minimum: $g'(x) = 2e^{2x} - 2ke^{-2x} = 0 \implies e^{4x} = k \implies x = \frac{1}{4} \ln k$.

At the minimum:

$$g = e^{2 \cdot \frac{1}{4} \ln k} + k e^{-2 \cdot \frac{1}{4} \ln k} = k^{1/2} + k \cdot k^{-1/2} = \sqrt{k} + \sqrt{k} = 2\sqrt{k}$$

Since $g''(x) = 4e^{2x} + 4ke^{-2x} > 0$, this is a minimum.

Range of g : $g(x) \geq 2\sqrt{k}$

Question 6 (Jun 2016, Q6)

Worked Solution

$C_1: y = \ln(4x - 7) + 18$; $C_2: y = a(x^2 + b)^{1/2}$. Point P has x -coordinate 2, and the gradients are equal at P .

Step 1: Find the y -coordinate of P on C_1 :

$$y = \ln(8 - 7) + 18 = \ln 1 + 18 = 18$$

So $P = (2, 18)$. Since P is on C_2 : $a(4 + b)^{1/2} = 18$ (1)

Step 2: Gradients.

$$C_1: \frac{dy}{dx} = \frac{4}{4x - 7}. \text{ At } x = 2: \text{ gradient} = \frac{4}{1} = 4.$$

$$C_2: \frac{dy}{dx} = a \cdot \frac{1}{2}(x^2 + b)^{-1/2} \cdot 2x = \frac{ax}{(x^2 + b)^{1/2}}.$$

$$\text{At } x = 2: \frac{2a}{(4 + b)^{1/2}} = 4, \text{ so } a = \frac{2(4 + b)^{1/2}}{1}, \text{ i.e. } 2a(4 + b)^{-1/2} = 4. \quad \dots (2)$$

Step 3: Solve simultaneously.

From (1): $a(4 + b)^{1/2} = 18$. From (2): $2a(4 + b)^{-1/2} = 4 \implies a(4 + b)^{-1/2} = 2$.

Divide (1) by the rearranged (2):

$$\frac{a(4 + b)^{1/2}}{a(4 + b)^{-1/2}} = \frac{18}{2} \implies (4 + b) = 9 \implies b = 5$$

From (2): $a = 2(4 + 5)^{1/2} \cdot \frac{1}{(4 + 5)^{1/2}} \cdot \frac{(4 + b)^{1/2}}{1}$; using $a(4 + 5)^{1/2} = 18$: $a \cdot 3 = 18 \implies a = 6$.

$$a = 6, b = 5$$

Question 7 (Jun 2017, Q9)

Worked Solution

(a) $y = \frac{px + q}{x^2 + 3}$. Show that the curve has two distinct stationary points for all non-zero p, q .

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{p(x^2 + 3) - (px + q) \cdot 2x}{(x^2 + 3)^2} = \frac{px^2 + 3p - 2px^2 - 2qx}{(x^2 + 3)^2} = \frac{-px^2 - 2qx + 3p}{(x^2 + 3)^2}$$

Set numerator to zero: $px^2 + 2qx - 3p = 0$.

Discriminant: $\Delta = 4q^2 + 12p^2 = 4(q^2 + 3p^2)$.

Since $p \neq 0$, we have $3p^2 > 0$, so $\Delta = 4(q^2 + 3p^2) > 0$ for all non-zero p, q .

Therefore there are always two distinct stationary points. □

(b) $y = e^{x^2}(ax^2 + b)$. Given $\frac{d^2y}{dx^2} = e^{x^2}(cx^4 + d)$, prove that $5a + 2b = 0$.

First derivative (product rule, with $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$):

$$\frac{dy}{dx} = 2xe^{x^2}(ax^2 + b) + e^{x^2}(2ax) = e^{x^2}(2ax^3 + 2bx + 2ax) = e^{x^2}(2ax^3 + (2a + 2b)x)$$

Second derivative (product rule again):

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2xe^{x^2}(2ax^3 + (2a + 2b)x) + e^{x^2}(6ax^2 + (2a + 2b)) \\ &= e^{x^2}[4ax^4 + (4a + 4b)x^2 + 6ax^2 + (2a + 2b)] = e^{x^2}[4ax^4 + (10a + 4b)x^2 + (2a + 2b)] \end{aligned}$$

This must equal $e^{x^2}(cx^4 + d)$, so the coefficient of x^2 must be zero:

$$10a + 4b = 0 \implies 5a + 2b = 0 \quad \square$$

Question 8 (OCR 4753, Jan 2010, Q8i,ii)

Worked Solution

$y = x \cos 3x$. The curve crosses the x -axis at O, P and Q.

(i) Find the exact coordinates of P and Q.

At crossings (other than O), $y = 0$ with $x \neq 0$, so $\cos 3x = 0$:

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \implies x = \frac{\pi}{6}, \frac{\pi}{2}, \dots$$

From the sketch, P and Q are the first two positive crossings:

$$P = \left(\frac{\pi}{6}, 0\right); \quad Q = \left(\frac{\pi}{2}, 0\right)$$

(ii) Find the exact gradient at P, and show turning points satisfy $x \tan 3x = \frac{1}{3}$.

Differentiate using the product rule:

$$\frac{dy}{dx} = \cos 3x + x \cdot (-3 \sin 3x) = \cos 3x - 3x \sin 3x$$

At P, $x = \frac{\pi}{6}$:

$$\frac{dy}{dx} = \cos \frac{\pi}{2} - 3 \cdot \frac{\pi}{6} \cdot \sin \frac{\pi}{2} = 0 - \frac{\pi}{2} \cdot 1 = -\frac{\pi}{2}$$

$$\text{Gradient at } P = -\frac{\pi}{2}$$

At turning points, $\frac{dy}{dx} = 0$:

$$\cos 3x - 3x \sin 3x = 0 \implies \cos 3x = 3x \sin 3x \implies \frac{1}{3} = \frac{\sin 3x}{\cos 3x} \cdot x = x \tan 3x$$

$$\therefore x \tan 3x = \frac{1}{3} \quad \square$$

End of Worked Solutions