Differential Equations Exam Questions (From OCR 4724)

Q1, (Jan 2006, Q8)

(i) Solve the differential equation

$$\frac{dy}{dx} = \frac{2-x}{y-3},$$

giving the particular solution that satisfies the condition $y = 4$ when $x = 5$. [5]

(ii) Show that this particular solution can be expressed in the form

$$(x-a)^2 + (y-b)^2 = k,$$

where the values of the constants a, b and k are to be stated. [3]

(iii) Hence sketch the graph of the particular solution, indicating clearly its main features. [3]

Q2, (Jan 2007, Q9)

(i) Find the general solution of the differential equation

$$\frac{\sec^2 y}{\cos^2(2x)} \frac{dy}{dx} = 2.$$ [7]

(ii) For the particular solution in which $y = \frac{1}{4}\pi$ when $x = 0$, find the value of y when $x = \frac{1}{6}\pi$. [3]

Q3, (Jun 2007, Q8)

The height, h metres, of a shrub t years after planting is given by the differential equation

$$\frac{dh}{dt} = \frac{6-h}{20}.$$

A shrub is planted when its height is 1 m.

(i) Show by integration that $t = 20\ln\left(\frac{5}{6-h}\right)$. [6]

(ii) How long after planting will the shrub reach a height of 2 m? [1]

(iii) Find the height of the shrub 10 years after planting. [2]

(iv) State the maximum possible height of the shrub. [1]

Q4, (Jan 2008, Q8)

Water flows out of a tank through a hole in the bottom and, at time t minutes, the depth of water in the tank is x metres. At any instant, the rate at which the depth of water in the tank is decreasing is proportional to the square root of the depth of water in the tank.

(i) Write down a differential equation which models this situation. [2]

(ii) When $t = 0$, $x = 2$; when $t = 5$, $x = 1$. Find t when $x = 0.5$, giving your answer correct to 1 decimal place. [6]

Q5, (Jan 2009, Q9)

A liquid is being heated in an oven maintained at a constant temperature of 160 °C. It may be assumed that the rate of increase of the temperature of the liquid at any particular time t minutes is proportional to $160 - \theta$, where θ °C is the temperature of the liquid at that time.

(i) Write down a differential equation connecting θ and t. **[2]**

When the liquid was placed in the oven, its temperature was 20 °C and 5 minutes later its temperature had risen to 65 °C.

(ii) Find the temperature of the liquid, correct to the nearest degree, after another 5 minutes. **[9]**

Q6, (Jan 2011, Q9)

Paraffin is stored in a tank with a horizontal base. At time t minutes, the depth of paraffin in the tank is x cm. When $t = 0$, $x = 72$. There is a tap in the side of the tank through which the paraffin can flow. When the tap is opened, the flow of the paraffin is modelled by the differential equation

$$\frac{dx}{dt} = -4(x-8)^{\frac{1}{3}}.$$

(i) How long does it take for the level of paraffin to fall from a depth of 72 cm to a depth of 35 cm? **[7]**

(ii) The tank is filled again to its original depth of 72 cm of paraffin and the tap is then opened. The paraffin flows out until it stops. How long does this take? **[3]**

Q7, (Jun 2014, Q10)

A container in the shape of an inverted cone of radius 3 metres and vertical height 4.5 metres is initially filled with liquid fertiliser. This fertiliser is released through a hole in the bottom of the container at a rate of 0.01 m^3 per second. At time t seconds the fertiliser remaining in the container forms an inverted cone of height h metres.

[The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]

(i) Show that $h^2 \frac{dh}{dt} = -\frac{9}{400\pi}$. **[5]**

(ii) Express h in terms of t. **[4]**

(iii) Find the time it takes to empty the container, giving your answer to the nearest minute. **[2]**

Q8, (Jun 2015, Q8)

In the year 2000 the population density, P, of a village was 100 people per km^2, and was increasing at the rate of 1 person per km^2 per year. The rate of increase of the population density is thought to be inversely proportional to the size of the population density. The time in years after the year 2000 is denoted by t.

(i) Write down a differential equation to model this situation, and solve it to express P in terms of t. **[6]**

(ii) In 2008 the population density of the village was 108 people per km^2 and in 2013 it was 128 people per km^2. Determine how well the model fits these figures. **[2]**

Q9, (Jun 2016, Q10)

(i) Express $\dfrac{16+5x-2x^2}{(x+1)^2(x+4)}$ in partial fractions. **[5]**

(ii) It is given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(16+5x-2x^2)y}{(x+1)^2(x+4)}$$

and that $y = \dfrac{1}{256}$ when $x = 0$. Find the exact value of y when $x = 2$. Give your answer in the form Ae^n. **[7]**

Q10, (Jun 2017, Q7)

The surface of a pond is covered by water lilies. The area of water lilies is denoted by A m^2. At $t = 0$, $A = 10$ and $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 0.48$. It is thought that eventually the lilies will cover the whole of the surface area of the pond. A biologist proposes that this situation is modelled by the differential equation

$$\left(\frac{1}{A} + \frac{1}{250-A}\right)\frac{\mathrm{d}A}{\mathrm{d}t} = k$$

where t is the time in days and k is a constant.

(i) Solve this differential equation to express A in terms of t and k. **[6]**

(ii) Find the value of k. **[1]**

(iii) Assuming the model is reliable, find the surface area of the pond. **[1]**

Q11, (Jun 2018, Q9)

When a container is partially filled with liquid to a depth of x centimetres, the volume $V\,\text{cm}^3$ of liquid in the container is given by the formula

$$V = (x+1)^3 - 1.$$

Initially the container is empty. Liquid is poured into the container so that the rate at which V increases is directly proportional to e^{-t}, where t is the time in seconds since the addition of liquid began. When $t = 2$, the rate at which V is increasing is $10\,\text{cm}^3\,\text{s}^{-1}$.

(i) Show that $V = 10e^2(1 - e^{-t})$. Hence find how long it takes for the depth of liquid in the container to reach 3 cm. **[8]**

(ii) Find the value that the depth of liquid approaches as t increases, giving your answer correct to 3 significant figures. **[3]**