Question 1 (Jan 2006, Q8)

Worked Solution

(i) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-x}{y-3}$, $y = 4$ when $x = 5$:

Separate:

$$(y-3)\,\mathrm{d}y = (2-x)\,\mathrm{d}x$$

$$\tfrac{1}{2}y^2 - 3y = 2x - \tfrac{1}{2}x^2 + C$$

When $x = 5$, $y = 4$: $\frac{1}{2}(16) - 12 = 10 - \frac{1}{2}(25) + C \Rightarrow 8 - 12 = 10 - 12.5 + C \Rightarrow -4 = -2.5 + C \Rightarrow C = -\frac{3}{2}$.

$\frac{1}{2}y^2 - 3y = 2x - \frac{1}{2}x^2 - \frac{3}{2}$

(ii) Show this can be written as $(x-a)^2 + (y-b)^2 = k$:

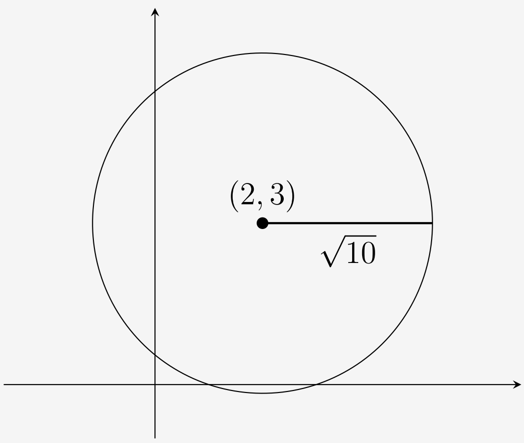
Multiply through by 2: $y^2 - 6y = 4x - x^2 - 3$.

Complete the square on both sides: $(y-3)^2 - 9 = -(x^2 - 4x) - 3 = -(x-2)^2 + 4 - 3 = -(x-2)^2 + 1$.

$(x-2)^2 + (y-3)^2 = 10$.

$a = 2$, $b = 3$, $k = 10$

(iii) Sketch: Circle, centre $(2, 3)$, radius $\sqrt{10}$.



Question 2 (Jan 2007, Q9)

Worked Solution

(i) General solution of $\dfrac{\sec^2 y}{\cos^2(2x)}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$:

Separate: $\sec^2 y\,\mathrm{d}y = 2\cos^2(2x)\,\mathrm{d}x$.

Use $\cos^2(2x) = \frac{1}{2}(1+\cos 4x)$:

$$\int \sec^2 y\,\mathrm{d}y = \int (1+\cos 4x)\,\mathrm{d}x$$

$$\tan y = x + \tfrac{1}{4}\sin 4x + c$$

$\tan y = x + \frac{1}{4}\sin 4x + c$

(ii) Find y when $x = \frac{1}{6}\pi$, given $y = \frac{1}{4}\pi$ when $x = 0$:

At $x = 0$, $y = \frac{\pi}{4}$: $\tan\frac{\pi}{4} = 0 + 0 + c \Rightarrow c = 1$.

At $x = \frac{\pi}{6}$:

$$\tan y = \tfrac{\pi}{6} + \tfrac{1}{4}\sin\left(\tfrac{2\pi}{3}\right) + 1 = \tfrac{\pi}{6} + \tfrac{\sqrt{3}}{8} + 1 \approx 0.5236 + 0.2165 + 1 = 1.7401\ldots$$

$$y = \arctan(1.7401\ldots) \approx 1.05 \text{ rad}$$

$y \approx 1.05$ radians

Question 3 (Jun 2007, Q8)

Worked Solution

$\frac{dh}{dt} = \frac{6-h}{20}$, $h = 1$ when $t = 0$.

(i) Show $t = 20\ln\left(\frac{5}{6-h}\right)$:

Separate:

$$\int \frac{1}{6-h}\,dh = \int \frac{1}{20}\,dt \implies -\ln(6-h) = \tfrac{t}{20} + c$$

At $t = 0$, $h = 1$: $c = -\ln 5$.

$$-\ln(6-h) = \tfrac{t}{20} - \ln 5 \implies \tfrac{t}{20} = \ln 5 - \ln(6-h) = \ln\left(\frac{5}{6-h}\right)$$

$$t = 20\ln\left(\frac{5}{6-h}\right) \quad \checkmark$$

(ii) Time to reach $h = 2$:

$$t = 20\ln\left(\frac{5}{4}\right) \approx 20 \times 0.2231 \approx 4.46 \text{ years}$$

$t \approx 4.46$ years (accept 4.5)

(iii) Height after $t = 10$ years:

$$10 = 20\ln\left(\frac{5}{6-h}\right) \implies \frac{5}{6-h} = e^{0.5} \implies 6-h = 5e^{-0.5} \implies h = 6-5e^{-0.5} \approx 6-3.033 = 2.97 \text{ m}$$

$h \approx 2.97$ m after 10 years

(iv) Maximum possible height:

As $t \to \infty$: $h \to 6$ m.

Maximum possible height = 6 m

Question 4 (Jan 2008, Q8)

Worked Solution

Water drains: depth x metres, rate of decrease proportional to $\sqrt{x}$.

(i) Differential equation:

$\frac{\mathrm{d}x}{\mathrm{d}t} = -k\sqrt{x}$ (where $k > 0$)

(ii) Find t when $x = 0.5$, given $x = 2$ at $t = 0$ and $x = 1$ at $t = 5$:

Separate:

$$\int x^{-1/2}\,\mathrm{d}x = \int (-k)\,\mathrm{d}t \implies 2\sqrt{x} = -kt + c$$

At $t = 0$, $x = 2$: $c = 2\sqrt{2}$.

At $t = 5$, $x = 1$: $2(1) = -5k + 2\sqrt{2} \Rightarrow 5k = 2\sqrt{2} - 2 \Rightarrow k = \frac{2(\sqrt{2}-1)}{5}$.

So $2\sqrt{x} = 2\sqrt{2} - kt$.

When $x = 0.5$: $2\sqrt{0.5} = \sqrt{2}$.

$$\sqrt{2} = 2\sqrt{2} - kt \implies kt = \sqrt{2} \implies t = \frac{\sqrt{2}}{k} = \frac{5\sqrt{2}}{2(\sqrt{2}-1)}$$

Rationalise: $t = \frac{5\sqrt{2}}{2(\sqrt{2}-1)} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{5\sqrt{2}(\sqrt{2}+1)}{2(1)} = \frac{5(2+\sqrt{2})}{2} \approx \frac{5 \times 3.414}{2} \approx$ 8.5

$t \approx 8.5$ minutes (1 d.p.)

Question 5 (Jan 2009, Q9)

Worked Solution

Oven at 160°C; rate of increase of liquid temperature proportional to $160 - \theta$.

(i) Differential equation:

$$\frac{d\theta}{dt} = k(160 - \theta) \text{ where } k > 0$$

(ii) Temperature after a further 5 minutes (i.e. at $t = 10$):

Separate:

$$-\ln(160 - \theta) = kt + c$$

At $t = 0$, $\theta = 20$: $c = -\ln 140$.

At $t = 5$, $\theta = 65$: $-\ln(95) = 5k - \ln 140 \Rightarrow 5k = \ln 140 - \ln 95 = \ln\frac{140}{95} = \ln\frac{28}{19}$.

$k = \frac{1}{5}\ln\frac{28}{19} \approx \frac{1}{5}(0.3857) = 0.07714$.

At $t = 10$:

$$-\ln(160 - \theta) = 10k - \ln 140 = 2\ln\frac{28}{19} - \ln 140$$

$$\ln(160 - \theta) = \ln 140 - 2\ln\frac{28}{19} = \ln\left(\frac{140 \times 19^2}{28^2}\right) = \ln\left(\frac{140 \times 361}{784}\right) = \ln(64.6\ldots)$$

Actually: $160 - \theta = 140\left(\frac{19}{28}\right)^2 = 140 \times \frac{361}{784} = \frac{140 \times 361}{784} = \frac{50540}{784} = 64.46\ldots$

Wait: $\frac{95}{140} = \frac{19}{28}$. At $t = 5$: $160 - 65 = 95$, and $160 - \theta_0 = 140$, so ratio $= \frac{95}{140} = \frac{19}{28}$.

Each additional 5 minutes multiplies $(160 - \theta)$ by $\frac{19}{28}$ (geometric):

At $t = 10$: $160 - \theta = 95 \times \frac{19}{28} = \frac{95 \times 19}{28} = \frac{1805}{28} = 64.46\ldots$

$\theta = 160 - 64.46 = 95.54\ldots \approx 96$°C.

Temperature after another 5 minutes ≈ 96°C

Question 6 (Jan 2011, Q9)

Worked Solution

$\frac{dx}{dt} = -4(x-8)^{1/3}$, $x = 72$ at $t = 0$.

(i) Time for x to fall from 72 to 35 cm:

Separate:

$$\int (x-8)^{-1/3}\,dx = \int -4\,dt \implies \tfrac{3}{2}(x-8)^{2/3} = -4t + c$$

At $t = 0$, $x = 72$: $\frac{3}{2}(64)^{2/3} = c = \frac{3}{2}(16) = 24$.

So $\frac{3}{2}(x-8)^{2/3} = 24 - 4t$.

When $x = 35$: $(35-8)^{2/3} = (27)^{2/3} = 9$.

$\frac{3}{2}(9) = 24 - 4t \Rightarrow 13.5 = 24 - 4t \Rightarrow 4t = 10.5 \Rightarrow t = \frac{21}{8} = 2.625$ min.

Time $= \frac{21}{8} \approx 2.63$ minutes

(ii) Time for flow to stop (starting from $x = 72$):

Flow stops when $x = 8$, i.e. $(x-8)^{2/3} = 0$.

$0 = 24 - 4t \Rightarrow t = 6$ min.

Flow stops after $t = 6$ minutes

Question 7 (Jun 2014, Q10)

Worked Solution

Inverted cone: radius 3 m, height 4.5 m. Similar triangles: $r/h = 3/4.5 = 2/3$, so $r = \frac{2h}{3}$.

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h = \frac{4\pi h^3}{27}$.

$\dfrac{dV}{dt} = -0.01\ \text{m}^3\text{s}^{-1}$.

(i) Show $h^2 \dfrac{dh}{dt} = -\dfrac{9}{400\pi}$:

$\dfrac{dV}{dh} = \dfrac{4\pi h^2}{9}$.

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{-0.01}{4\pi h^2/9} = \frac{-0.09}{4\pi h^2}$$

$$h^2\frac{dh}{dt} = \frac{-0.09}{4\pi} = \frac{-9}{400\pi} \quad \checkmark$$

(ii) Express h in terms of t:

Separate:

$$\int h^2\,dh = \int \frac{-9}{400\pi}\,dt \implies \frac{h^3}{3} = \frac{-9t}{400\pi} + c$$

At $t = 0$, $h = 4.5 = \frac{9}{2}$: $c = \frac{1}{3}\left(\frac{9}{2}\right)^3 = \frac{729}{24} = \frac{243}{8}$.

$$\frac{h^3}{3} = \frac{243}{8} - \frac{9t}{400\pi} \implies h^3 = \frac{729}{8} - \frac{27t}{400\pi}$$

$$h = \left(\frac{729}{8} - \frac{27t}{400\pi}\right)^{1/3}$$

$$h = \left(\frac{729}{8} - \frac{27t}{400\pi}\right)^{1/3}$$

(iii) Time to empty ($h = 0$), to nearest minute:

$0 = \dfrac{729}{8} - \dfrac{27t}{400\pi} \Rightarrow t = \dfrac{729 \times 400\pi}{8 \times 27} = \dfrac{729 \times 400\pi}{216} = \dfrac{729\pi}{0.54} = \dfrac{400\pi \times 729}{216} = \dfrac{400\pi \times 27}{8} = \dfrac{10800\pi}{8} = 1350\pi \approx 4241\ \text{s} \approx 71\ \text{min}$.

Time to empty ≈ 71 minutes

Question 8 (Jun 2015, Q8)

Worked Solution

Population density P, rate of increase inversely proportional to P: $\frac{dP}{dt} = \frac{k}{P}$.

At $t = 0$, $P = 100$ and $\frac{dP}{dt} = 1$: $k = 100$.

(i) Solve and express P in terms of t:

$$P\,dP = k\,dt \implies \frac{P^2}{2} = kt + c$$

At $t = 0$, $P = 100$: $c = 5000$.

$\frac{P^2}{2} = 100t + 5000 \Rightarrow P^2 = 200t + 10000$.

$P = \sqrt{200t + 10000} = \sqrt{200(50 + t)} = 10\sqrt{2(50 + t)}$

(ii) How well does the model fit?

In 2008 ($t = 8$): $P = \sqrt{200(8) + 10000} = \sqrt{11600} \approx 107.7 \approx 108$. Model gives ≈ 108; actual is 108. Good fit.

In 2013 ($t = 13$): $P = \sqrt{200(13) + 10000} = \sqrt{12600} \approx 112.2$. Actual is 128. Poor fit – model underestimates significantly.

Model fits well in 2008 (≈ 108 predicted vs 108 actual) but poorly in 2013 (≈ 112 predicted vs 128 actual).

Question 9 (Jun 2016, Q10)

Worked Solution

(i) Partial fractions of $\frac{16+5x-2x^2}{(x+1)^2(x+4)}$:

$$\frac{16+5x-2x^2}{(x+1)^2(x+4)} = \frac{A}{x+4} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$16 + 5x - 2x^2 = A(x+1)^2 + B(x+1)(x+4) + C(x+4)$.

$x = -4$: $16 - 20 - 32 = 9A \Rightarrow -36 = 9A \Rightarrow A = -4$.

$x = -1$: $16 - 5 - 2 = 3C \Rightarrow 9 = 3C \Rightarrow C = 3$.

$x = 0$: $16 = A(1) + B(4) + C(4) = -4 + 4B + 12 \Rightarrow 4B = 8 \Rightarrow B = 2$.

$$\frac{-4}{x+4} + \frac{2}{x+1} + \frac{3}{(x+1)^2}$$

(ii) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(16+5x-2x^2)y}{(x+1)^2(x+4)}$, $y = \frac{1}{256}$ when $x = 0$; find y when $x = 2$:

Separate:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \left(\frac{-4}{x+4} + \frac{2}{x+1} + \frac{3}{(x+1)^2}\right)\mathrm{d}x$$

$$\ln y = -4\ln(x+4) + 2\ln(x+1) - \frac{3}{x+1} + c$$

At $x = 0$, $y = \frac{1}{256}$:

$$\ln\tfrac{1}{256} = -4\ln 4 + 2\ln 1 - 3 + c = -4\ln 4 - 3 + c$$

$$-\ln 256 = -4\ln 4 - 3 + c$$

Since $\ln 256 = \ln 4^4 = 4\ln 4$: $-4\ln 4 = -4\ln 4 - 3 + c \Rightarrow c = 3$.

At $x = 2$:

$$\ln y = -4\ln 6 + 2\ln 3 - \frac{3}{3} + 3 = -4\ln 6 + 2\ln 3 + 2$$

$$\ln y = \ln\left(\frac{3^2}{6^4}\right) + 2 = \ln\left(\frac{9}{1296}\right) + 2 = \ln\left(\frac{1}{144}\right) + 2$$

$$y = \frac{1}{144}e^2 = \frac{e^2}{144}$$

$$y = \frac{e^2}{144}$$

Question 10 (Jun 2017, Q7)

Worked Solution

$\left(\frac{1}{A}+\frac{1}{250-A}\right)\frac{\mathrm{d}A}{\mathrm{d}t}=k$, with $A=10$ and $\frac{\mathrm{d}A}{\mathrm{d}t}=0.48$ at $t=0$.

(i) Solve to express A in terms of t and k:

Separate:

$$\int\left(\frac{1}{A}+\frac{1}{250-A}\right)\mathrm{d}A=\int k\,\mathrm{d}t$$

$$\ln A-\ln(250-A)=kt+c \implies \ln\left(\frac{A}{250-A}\right)=kt+c$$

At $t=0,\ A=10$: $c=\ln\frac{10}{240}=\ln\frac{1}{24}=-\ln 24$.

$$\ln\left(\frac{A}{250-A}\right)=kt-\ln 24$$

$$\frac{A}{250-A}=\frac{e^{kt}}{24} \implies 24A=e^{kt}(250-A) \implies A(24+e^{kt})=250e^{kt}$$

$$A=\frac{250e^{kt}}{24+e^{kt}}$$

(ii) Find k:

From $\frac{\mathrm{d}A}{\mathrm{d}t}=0.48$ at $t=0,\ A=10$:

$\left(\frac{1}{10}+\frac{1}{240}\right)(0.48)=k \Rightarrow \frac{250}{2400}(0.48)=k \Rightarrow k=\frac{250\times 0.48}{2400}=\frac{120}{2400}=0.05.$

$k=0.05$

(iii) Surface area of pond:

As $t\to\infty$, $A\to 250$ m^2 (from denominator $\to e^{kt}$, so $A\to 250$).

Surface area of pond $= 250$ m^2

Question 11 (Jun 2018, Q9)

Worked Solution

$V = (x+1)^3 - 1$. Rate $\dfrac{dV}{dt} \propto e^{-t}$; at $t = 2$, $\dfrac{dV}{dt} = 10$.

(i) Show $V = 10e^2(1 - e^{-t})$ and find t when $x = 3$ cm:

$\dfrac{dV}{dt} = ke^{-t}$. At $t = 2$: $ke^{-2} = 10 \Rightarrow k = 10e^2$.

Integrate: $V = -10e^2e^{-t} + c$.

At $t = 0$, $V = 0$ (container empty, $x = 0$): $0 = -10e^2 + c \Rightarrow c = 10e^2$.

$$V = 10e^2 - 10e^2e^{-t} = 10e^2(1 - e^{-t}) \quad \checkmark$$

When $x = 3$: $V = (3+1)^3 - 1 = 64 - 1 = 63$.

$$63 = 10e^2(1 - e^{-t}) \implies 1 - e^{-t} = \frac{63}{10e^2} \implies e^{-t} = 1 - \frac{63}{10e^2}$$
$$-t = \ln\left(1 - \frac{63}{10e^2}\right) \implies t = -\ln\left(1 - \frac{63}{10e^2}\right)$$

Numerically: $10e^2 = 73.891$, so $\dfrac{63}{73.891} = 0.8526$.

$1 - 0.8526 = 0.1474$. $t = -\ln(0.1474) \approx 1.914$.

$t \approx 1.91$ to 1.915 seconds

(ii) Depth approached as $t \to \infty$:

As $t \to \infty$: $e^{-t} \to 0$, so $V \to 10e^2$.

$(x+1)^3 - 1 = 10e^2 \Rightarrow (x+1)^3 = 1 + 10e^2 \Rightarrow x = (1 + 10e^2)^{1/3} - 1$.

$1 + 10e^2 = 74.891$, $(74.891)^{1/3} = 4.2151\ldots$, so $x = 3.215\ldots$.

Depth approaches $x \approx 3.22$ cm (3 s.f.)

End of Worked Solutions