Differential Equations (Sheet 2)

Q1.

(a) Express $\frac{2}{4-y^2}$ in partial fractions.

(3)

(b) Hence obtain the solution of

$$2\cot x \frac{dy}{dx} = (4 - y^2)$$

for which $y = 0$ at $x = \frac{\pi}{3}$, giving your answer in the form $\sec^2 x = g(y)$.

(8)

(Total 11 marks)

Q2.

The height above ground, H metres, of a passenger on a roller coaster can be modelled by the differential equation

$$\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$$

where t is the time, in seconds, from the start of the ride.

Given that the passenger is 5 m above the ground at the start of the ride,

(a) show that $H = 5e^{0.1\sin(0.25t)}$

(5)

(b) State the maximum height of the passenger above the ground.

(1)

The passenger reaches the maximum height, for the second time, T seconds after the start of the ride.

(c) Find the value of T.

(2)

(Total for question = 8 marks)

Q3.

Water is being heated in a kettle. At time t seconds, the temperature of the water is θ °C.

The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leqslant 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$,

(a) solve this differential equation to show that

$$\theta = 120 - 100e^{-\lambda t}$$

(8)

When the temperature of the water reaches 100 °C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off.

(3)

(Total 11 marks)

Q4.

Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 $cm^3\ s^{-1}$ and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$\frac{dh}{dt} = 0.4 - k\sqrt{h}$, where k is a positive constant.

(3)

When $h = 25$, water is leaking out of the hole at 400 $cm^3\ s^{-1}$.

(b) Show that $k = 0.02$

(1)

(c) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h},$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(d) find the exact value of $\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh.$

(6)

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

(Total 13 marks)

Q5.

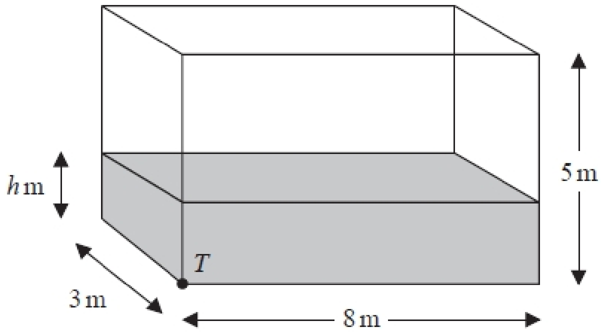

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point *T* at the bottom of the tank, as shown in Figure 5.

At time *t* minutes after the tap has been opened

- the depth of water in the tank is *h* metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\,e^{-kt}$$

where *A*, *B* and *k* are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

(Total for question = 12 marks)

Q6.

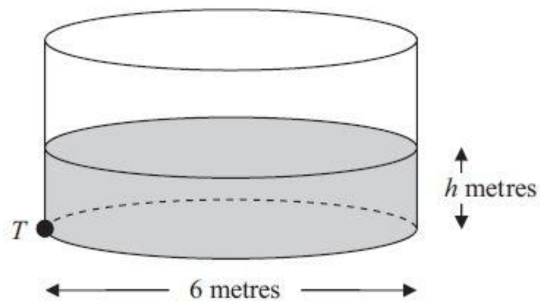

Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time *t* minutes, the depth of the water in the tank is *h* metres. There is a tap at a point *T* at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that *t* minutes after the tap has been opened

$$75\frac{dh}{dt} = (4 - 5h)$$

(5)

When $t = 0$, $h = 0.2$

(b) Find the value of *t* when $h = 0.5$

(6)
(Total 11 marks)

Q7.
A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP,$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(a) solve the differential equation, giving P in terms of P_0, k and t.

(4)

Given also that $k = 2.5$,

(b) find the time taken, to the nearest minute, for the population to reach $2P_0$.

(3)

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(c) solve the second differential equation, giving P in terms of P_0, λ and t.

(4)

Given also that $\lambda = 2.5$,

(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model.

(3)

(Total 14 marks)

Q8.

In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the experiment. It is known that at time t minutes, the rate of increase of the mass of waste products, in kg per minute, is k times the mass of unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{dx}{dt} = k(M - x), \text{ where } M \text{ is a constant.}$$

(a) Explain, in the context of the problem, what $\frac{dx}{dt}$ and M represent.

(2)

Given that initially the mass of waste products is zero,

(b) solve the differential equation, expressing x in terms of k, M and t.

(6)

Given also that $x = \frac{1}{2}M$ when $t = \ln 4$,

(b) find the value of x when $t = \ln 9$, expressing x in terms of M, in its simplest form.

(4)

(Total 12 marks)

Q9.

The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt - 1)(5000 - N)}{t} \qquad t > 0, \quad 0 < N < 5000$$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(a) By solving the differential equation, show that

$$N = 5000 - Ate^{-kt}$$

where A is a positive constant.

(5)

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(b) Find the exact value of the constant A and the exact value of the constant k.

(4)

(c) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish.

(1)

(Total 10 marks)

Q10.

(a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{dh}{4 - \sqrt{h}} = -8\ln|4 - \sqrt{h}| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

(Total for question = 15 marks)