Question 1

Worked Solution

(a) Partial fractions of $\frac{2}{4-y^2}$:

$$\frac{2}{4-y^2} = \frac{2}{(2-y)(2+y)} = \frac{A}{2-y} + \frac{B}{2+y}$$

$2 = A(2+y) + B(2-y)$. Let $y = 2$: $2 = 4B \Rightarrow B = \frac{1}{2}$. Let $y = -2$: $2 = 4A \Rightarrow A = \frac{1}{2}$.

$$\frac{2}{4-y^2} = \frac{1/2}{2-y} + \frac{1/2}{2+y}$$

(b) Solve $2\cot x\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - y^2$, $y = 0$ when $x = \frac{\pi}{3}$:

Separate variables:

$$\frac{2}{4-y^2}\,\mathrm{d}y = \frac{1}{\cot x}\,\mathrm{d}x = \tan x\,\mathrm{d}x$$

$$\int \left(\frac{1/2}{2-y} + \frac{1/2}{2+y}\right)\mathrm{d}y = \int \tan x\,\mathrm{d}x$$

$$-\tfrac{1}{2}\ln(2-y) + \tfrac{1}{2}\ln(2+y) = \ln(\sec x) + c$$

$$\tfrac{1}{2}\ln\left(\frac{2+y}{2-y}\right) = \ln(\sec x) + c$$

Apply $y = 0$, $x = \frac{\pi}{3}$: $\sec\frac{\pi}{3} = 2$, so $\frac{1}{2}\ln(1) = \ln 2 + c \Rightarrow c = -\ln 2$.

$$\tfrac{1}{2}\ln\left(\frac{2+y}{2-y}\right) = \ln(\sec x) - \ln 2 = \ln\left(\frac{\sec x}{2}\right)$$

$$\ln\left(\frac{2+y}{2-y}\right) = 2\ln\left(\frac{\sec x}{2}\right) = \ln\left(\frac{\sec^2 x}{4}\right)$$

$$\frac{2+y}{2-y} = \frac{\sec^2 x}{4}$$

Rearrange (cross-multiply and solve for $\sec^2 x$): $4(2+y) = \sec^2 x(2-y)$, so $\sec^2 x = \frac{4(2+y)}{2-y} = \frac{8+4y}{2-y}$.

$$\sec^2 x = \frac{8+4y}{2-y}$$

Question 2

Worked Solution

$\dfrac{\mathrm{d}H}{\mathrm{d}t} = \dfrac{H\cos(0.25t)}{40}, \quad H = 5 \text{ when } t = 0.$

(a) Show $H = 5e^{0.1\sin(0.25t)}$:

Separate variables:

$$\int \frac{1}{H}\,\mathrm{d}H = \int \frac{\cos(0.25t)}{40}\,\mathrm{d}t$$

$$\ln H = \frac{\sin(0.25t)}{40 \times 0.25} + c = \frac{\sin(0.25t)}{10} + c$$

When $t = 0$, $H = 5$: $\ln 5 = 0 + c$, so $c = \ln 5$.

$$\ln\left(\frac{H}{5}\right) = \frac{\sin(0.25t)}{10} \implies H = 5e^{0.1\sin(0.25t)} \quad \checkmark$$

(b) Maximum height:

Maximum when $\sin(0.25t) = 1$: $H_{\max} = 5e^{0.1}$.

Maximum height $= 5e^{0.1} \approx 5.53$ m

(c) Value of T (second time maximum reached):

Maximum occurs when $\sin(0.25t) = 1$, i.e. $0.25t = \frac{\pi}{2} + 2n\pi$.

First time: $t = 2\pi \approx 6.28$ s.

Second time: $0.25t = \frac{\pi}{2} + 2\pi \Rightarrow 0.25t = \frac{5\pi}{2} \Rightarrow t = 10\pi$.

$T = 10\pi \approx 31.4$ s

Question 3

Worked Solution

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(120 - \theta)$, $\theta \leq 100$, $\theta = 20$ when $t = 0$.

(a) Show $\theta = 120 - 100e^{-\lambda t}$:

Separate variables:

$$\int \frac{1}{120 - \theta}\,\mathrm{d}\theta = \int \lambda\,\mathrm{d}t$$
$$-\ln(120 - \theta) = \lambda t + c$$

When $t = 0$, $\theta = 20$: $-\ln(100) = c$.

$$-\ln(120 - \theta) = \lambda t - \ln 100 \implies \ln\left(\frac{120 - \theta}{100}\right) = -\lambda t$$
$$120 - \theta = 100e^{-\lambda t} \implies \theta = 120 - 100e^{-\lambda t} \quad \checkmark$$

(b) Time to reach $100°$C with $\lambda = 0.01$:

$$100 = 120 - 100e^{-0.01t} \implies 100e^{-0.01t} = 20 \implies e^{-0.01t} = 0.2$$
$$-0.01t = \ln(0.2) = -\ln 5 \implies t = 100\ln 5 \approx 160.94\ldots$$

Kettle switches off at $t = 161$ s (nearest second)

Question 4

Worked Solution

Cylinder area = 4000 cm^2, inflow = 1600 cm^{3}s^{-1}, outflow = $c\sqrt{h}$ cm^{3}s^{-1}.

(a) Show $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - k\sqrt{h}$**:**

Net rate of volume change: $\frac{\mathrm{d}V}{\mathrm{d}t} = 1600 - c\sqrt{h}$.

Since $V = 4000h$: $\frac{\mathrm{d}V}{\mathrm{d}h} = 4000$, so $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1600 - c\sqrt{h}}{4000} = 0.4 - \frac{c}{4000}\sqrt{h} = 0.4 - k\sqrt{h}$. ✓

(b) Show $k = 0.02$**:**

When $h = 25$, outflow = 400 cm^{3}s^{-1}: $c\sqrt{25} = 400 \Rightarrow 5c = 400 \Rightarrow c = 80$. $k = \frac{c}{4000} = \frac{80}{4000} = 0.02$. ✓

(c) Separate variables to show time $= \int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h$**:**

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.4 - 0.02\sqrt{h} \implies \mathrm{d}t = \frac{\mathrm{d}h}{0.4 - 0.02\sqrt{h}} = \frac{\mathrm{d}h}{0.02(20 - \sqrt{h})} = \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h \quad ✓$$

(d) Evaluate $\int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h$ **using** $h = (20 - x)^2$**:**

$\frac{\mathrm{d}h}{\mathrm{d}x} = -2(20 - x)$, so $\mathrm{d}h = -2(20 - x)\,\mathrm{d}x$.

Also $\sqrt{h} = 20 - x$ (since $20 - x > 0$), so $20 - \sqrt{h} = x$.

Limits: $h = 0 \Rightarrow x = 20$; $h = 100 \Rightarrow 20 - x = 10 \Rightarrow x = 10$.

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h = \int_{20}^{10} \frac{50}{x} \cdot (-2)(20 - x)\,\mathrm{d}x = \int_{10}^{20} \frac{100(20 - x)}{x}\,\mathrm{d}x$$

$$= 100\int_{10}^{20} \left(\frac{20}{x} - 1\right)\mathrm{d}x = 100\Big[20\ln x - x\Big]_{10}^{20}$$

$$= 100\Big[(20\ln 20 - 20) - (20\ln 10 - 10)\Big] = 100\Big[20\ln 2 - 10\Big] = 2000\ln 2 - 1000$$

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\,\mathrm{d}h = 2000\ln 2 - 1000$$

(e) Time to fill:

$2000\ln 2 - 1000 = 386.29\ldots$ s = 6 min 26 s.

Time = 6 minutes and 26 seconds

Question 5

Worked Solution

Cuboid tank: base $8\,\text{m}\times 3\,\text{m}$, height 5 m. Inflow $0.48\ \text{m}^3\,\text{min}^{-1}$, outflow $0.1h\ \text{m}^3\,\text{min}^{-1}$.

(a) Show $1200\,\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$**:**

$V = 24h$ (base area $= 24\ \text{m}^2$), so $\frac{\mathrm{d}V}{\mathrm{d}h} = 24$.

Net rate: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.48 - 0.1h$.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{0.48 - 0.1h}{24} \implies 24\,\frac{\mathrm{d}h}{\mathrm{d}t} = 0.48 - 0.1h$$

Multiply by 50: $1200\,\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$. ✓

(b) Solve with $h = 2$ when $t = 0$:

Separate variables:

$$\int \frac{1200}{24 - 5h}\,\mathrm{d}h = \int \mathrm{d}t \implies -240\ln(24 - 5h) = t + C$$

When $t = 0$, $h = 2$: $C = -240\ln(14)$.

$$t = 240\ln(14) - 240\ln(24 - 5h) = 240\ln\left(\frac{14}{24 - 5h}\right)$$

$$e^{t/240} = \frac{14}{24 - 5h} \implies 24 - 5h = 14e^{-t/240} \implies h = \frac{24 - 14e^{-t/240}}{5}$$

$$h = 4.8 - 2.8e^{-t/240}$$

$h = 4.8 - 2.8e^{-t/240}$, i.e. $A = 4.8$, $B = -2.8$, $k = \frac{1}{240}$

(c) Will the tank ever become full?

As $t \to \infty$: $h \to 4.8$ m. Since the tank height is 5 m and $h \to 4.8 < 5$, the tank never becomes full. The equilibrium depth is 4.8 m (where inflow = outflow).

No – the depth approaches a limit of 4.8 m, which is less than the tank height of 5 m.

Question 6

Worked Solution

Cylindrical tank: diameter 6 m, so radius = 3 m, cross-sectional area = 9π m^2.

Inflow = 0.48π m^3 min^{-1}, outflow = $0.6\pi h$ m^3 min^{-1}.

(a) Show $75\,\dfrac{dh}{dt} = 4 - 5h$:

$V = 9\pi h$, so $\dfrac{dV}{dh} = 9\pi$.

$\dfrac{dV}{dt} = 0.48\pi - 0.6\pi h$.

$$9\pi\,\frac{dh}{dt} = 0.48\pi - 0.6\pi h \implies 9\,\frac{dh}{dt} = 0.48 - 0.6h$$

Multiply by $\frac{75}{9}$... or divide by 0.12: multiply both sides by $\frac{25}{3}$:

$9\times\frac{25}{3}\,\dfrac{dh}{dt} = (0.48-0.6h)\times\frac{25}{3}$. Actually, divide $0.48-0.6h$ by 0.12: $\dfrac{0.48-0.6h}{0.12} = 4-5h$. Divide 9π by $0.12\pi = \frac{9}{0.12} = 75$. So:

$$75\,\frac{dh}{dt} = 4 - 5h \quad \checkmark$$

(b) Find t when $h = 0.5$, given $h = 0.2$ when $t = 0$:

Separate variables:

$$\int \frac{75}{4-5h}\,dh = \int dt \implies -15\ln(4-5h) = t + C$$

When $t = 0$, $h = 0.2$: $C = -15\ln(4-1) = -15\ln 3$.

$$t = -15\ln(4-5h) + 15\ln 3 = 15\ln\left(\frac{3}{4-5h}\right)$$

When $h = 0.5$: $4 - 5(0.5) = 1.5$.

$$t = 15\ln\left(\frac{3}{1.5}\right) = 15\ln 2$$

$t = 15\ln 2 \approx 10.4$ minutes

Question 7

Worked Solution

(a) Solve $\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP$**,** $P = P_0$ **at** $t = 0$**:**

$$\int \frac{1}{P}\,\mathrm{d}P = \int k\,\mathrm{d}t \implies \ln P = kt + c$$

At $t = 0$: $c = \ln P_0$.

$$P = P_0 e^{kt}$$

(b) Time to reach $2P_0$ **with** $k = 2.5$**:**

$2P_0 = P_0 e^{2.5t} \implies e^{2.5t} = 2 \implies t = \dfrac{\ln 2}{2.5} \approx 0.2773$ days.

In minutes: $0.2773 \times 24 \times 60 \approx 399$ min.

Time ≈ 399 minutes (6 hr 39 min)

(c) Solve $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \lambda P \cos \lambda t$**:**

$$\int \frac{1}{P}\,\mathrm{d}P = \int \lambda \cos \lambda t\,\mathrm{d}t \implies \ln P = \sin \lambda t + c$$

At $t = 0$: $c = \ln P_0$.

$$P = P_0 e^{\sin \lambda t}$$

(d) Time to reach $2P_0$ **with** $\lambda = 2.5$**:**

$P_0 e^{\sin 2.5t} = 2P_0 \implies \sin 2.5t = \ln 2$.

$2.5t = \arcsin(\ln 2) = \arcsin(0.6931\ldots) = 0.7726\ldots$ rad.

$t = \dfrac{0.7726}{2.5} \approx 0.3090$ days $= 0.3090 \times 24 \times 60 \approx 445$ min ≈ 7 hr 25 min.

Checking: mark scheme gives $t = \frac{1}{2.5}\arcsin(\ln 2) \approx 0.3063$ days $= 441$ min.

$\arcsin(\ln 2) = \arcsin(0.69315) = 0.76597\ldots$ rad.

$t = 0.76597/2.5 = 0.30639$ days $\times 1440 = 441.2$ min.

Time ≈ 441 minutes (7 hr 21 min)

Question 8

Worked Solution

$\dfrac{dx}{dt} = k(M - x)$, $x = 0$ at $t = 0$.

(a) Interpretations:

$\dfrac{dx}{dt}$ is the rate of increase of the mass of waste products (kg min^{-1}).

M is the total (constant) mass of unburned fuel plus waste products, i.e. the initial mass of unburned fuel.

(b) Solve for x in terms of k, M, t:

$$\int \frac{1}{M-x}\,dx = \int k\,dt \implies -\ln(M-x) = kt + c$$

At $t = 0$, $x = 0$: $c = -\ln M$.

$$-\ln(M-x) = kt - \ln M \implies \ln\left(\frac{M}{M-x}\right) = kt \implies M - x = Me^{-kt}$$

$x = M(1 - e^{-kt})$

(c) Find x when $t = \ln 9$, given $x = \frac{1}{2}M$ when $t = \ln 4$:

From $x = \frac{1}{2}M$ at $t = \ln 4$:

$$\tfrac{1}{2}M = M(1 - e^{-k\ln 4}) \implies \tfrac{1}{2} = 1 - 4^{-k} \implies 4^{-k} = \tfrac{1}{2} \implies k = \tfrac{1}{2}$$

At $t = \ln 9$:

$$x = M\left(1 - e^{-\frac{1}{2}\ln 9}\right) = M\left(1 - 9^{-1/2}\right) = M\left(1 - \tfrac{1}{3}\right) = \tfrac{2}{3}M$$

$x = \dfrac{2}{3}M$

Question 9

Worked Solution

$\dfrac{\mathrm{d}N}{\mathrm{d}t} = \dfrac{(kt-1)(5000-N)}{t}$, $t > 0$, $0 < N < 5000$.

(a) Show $N = 5000 - Ate^{-kt}$:

Separate variables:

$$\int \frac{1}{5000-N}\,\mathrm{d}N = \int \frac{kt-1}{t}\,\mathrm{d}t = \int \left(k - \frac{1}{t}\right)\mathrm{d}t$$

$$-\ln(5000-N) = kt - \ln t + c$$

$$\ln(5000-N) = -kt + \ln t - c = \ln t - kt - c$$

$$5000 - N = e^{-c} \cdot te^{-kt} = Ate^{-kt}$$

where $A = e^{-c} > 0$.

$N = 5000 - Ate^{-kt}$ ✓

(b) Find A and k exactly:

$t = 1$, $N = 1200$: $1200 = 5000 - Ae^{-k} \Rightarrow Ae^{-k} = 3800$.

$t = 2$, $N = 1800$: $1800 = 5000 - 2Ae^{-2k} \Rightarrow 2Ae^{-2k} = 3200 \Rightarrow Ae^{-2k} = 1600$.

Dividing: $\dfrac{Ae^{-k}}{Ae^{-2k}} = \dfrac{3800}{1600} \Rightarrow e^k = \dfrac{19}{8} \Rightarrow k = \ln\dfrac{19}{8}$.

Then $A = 3800e^k = 3800 \times \dfrac{19}{8} = \dfrac{3800 \times 19}{8} = 9025$.

$A = 9025, \quad k = \ln\left(\dfrac{19}{8}\right)$

(c) Number of fish after 5 years:

$$N = 5000 - 9025 \times 5 \times e^{-5\ln(19/8)} = 5000 - 45125 \times \left(\frac{8}{19}\right)^5$$

$$\left(\frac{8}{19}\right)^5 = \frac{32768}{2476099} \approx 0.01323\ldots$$

$$N \approx 5000 - 45125 \times 0.01323 = 5000 - 596.9\ldots \approx 4403$$

$N \approx 4400$ fish (nearest hundred)

Question 10

Worked Solution

(a) Show $\int \frac{\mathrm{d}h}{4-\sqrt{h}} = -8\ln|4-\sqrt{h}| - 2\sqrt{h} + k$**:**

Let $u = 4 - \sqrt{h}$, so $\sqrt{h} = 4 - u$ and $h = (4-u)^2$, giving $\mathrm{d}h = -2(4-u)\,\mathrm{d}u$.

$$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = \int \frac{-2(4-u)}{u}\,\mathrm{d}u = \int \left(-\frac{8}{u} + 2\right)\mathrm{d}u = -8\ln|u| + 2u + c$$
$$= -8\ln|4-\sqrt{h}| + 2(4-\sqrt{h}) + c = -8\ln|4-\sqrt{h}| - 2\sqrt{h} + k \quad \checkmark$$

(absorbing the constant 8 into k)

(b) Range of heights: $0 < h < 16$**:**

$\frac{\mathrm{d}h}{\mathrm{d}t} = 0$ when $4 - \sqrt{h} = 0$, i.e. $h = 16$. Since $t^{0.25} > 0$, $\frac{\mathrm{d}h}{\mathrm{d}t} > 0$ for $h < 16$, so height increases towards but never reaches 16 m.

> Range: $0 < h < 16$ metres

(c) Time to grow from $h = 1$ **to** $h = 12$ **m:**

Separate variables:

$$\frac{\mathrm{d}h}{4-\sqrt{h}} = \frac{t^{0.25}}{20}\,\mathrm{d}t$$

Integrate both sides:

$$-8\ln(4-\sqrt{h}) - 2\sqrt{h} = \frac{t^{1.25}}{25} + C$$

At $t = 0$, $h = 1$: $-8\ln(4-1) - 2(1) = C \Rightarrow C = -8\ln 3 - 2$. So $k = 2 + 8\ln 3$.

At $h = 12$:

$$-8\ln(4-\sqrt{12}) - 2\sqrt{12} + (2 + 8\ln 3) = \frac{t^{1.25}}{25}$$
$$-8\ln(4-2\sqrt{3}) - 4\sqrt{3} + 2 + 8\ln 3 = \frac{t^{1.25}}{25}$$

Compute numerically: $\sqrt{12} = 3.4641$, $4 - \sqrt{12} = 0.5359$.

$-8\ln(0.5359) - 2(3.4641) + 2 + 8\ln 3 = -8(-0.6236) - 6.9282 + 2 + 8(1.0986) = 4.9888 - 6.9282 + 2 + 8.7889 = 8.8495$

So $t^{1.25} = 25 \times 8.8495 = 221.24$.

$t = 221.24^{1/1.25} = 221.24^{0.8} \approx 75.2$ years.

> $t \approx 75.2$ years (3 s.f.)

End of Worked Solutions