Question 1 (Jan 2007, Q4)

Worked Solution

$x = (4t+9)^{1/2}, \quad y = 6e^{\frac{1}{2}x+1}$.

(i) Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}(4t+9)^{-1/2} \cdot 4 = 2(4t+9)^{-1/2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6 \cdot \tfrac{1}{2}e^{\frac{1}{2}x+1} = 3e^{\frac{1}{2}x+1}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(4t+9)^{-1/2}, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3e^{\frac{1}{2}x+1}$$

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $t = 4$:

When $t = 4$: $x = (16+9)^{1/2} = 5$.

By chain rule: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3e^{\frac{1}{2}(5)+1} \times 2(4\times 4+9)^{-1/2} = 3e^{3.5} \times 2(25)^{-1/2} = 3e^{3.5} \times \frac{2}{5}$$

$$= \frac{6}{5}e^{3.5} = 1.2e^{3.5} \approx 39.7$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} \approx 39.7 \text{ (3 s.f.)}$$

Question 2 (Jan 2008, Q4)

Worked Solution

$V = (h^6 + 16)^{1/2} - 4.$

(i) Find $\frac{dV}{dh}$ when $h = 2$:

$$\frac{dV}{dh} = \frac{1}{2}(h^6 + 16)^{-1/2} \cdot 6h^5 = \frac{3h^5}{(h^6 + 16)^{1/2}}$$

When $h = 2$: $h^6 = 64$, so $(h^6 + 16)^{1/2} = (80)^{1/2}$.

$$\frac{dV}{dh} = \frac{3(32)}{(80)^{1/2}} = \frac{96}{\sqrt{80}} = \frac{96}{4\sqrt{5}} = \frac{24}{\sqrt{5}} \approx 10.7$$

$\frac{dV}{dh} \approx 10.7$ when $h = 2$

(ii) Rate of increase of height when $h = 2$, given $\frac{dV}{dt} = 8$ m^{3}hr^{-1}:

$$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = \frac{8}{10.733\ldots} \approx 0.75 \text{ m hr}^{-1}$$

$\frac{dh}{dt} \approx 0.75$ m hr^{-1} (2 s.f.)

Question 3 (Jan 2010, Q7a)

Worked Solution

Oil patch: circular area A increasing at 250 m^2hr^{-1}. Find $\frac{\mathrm{d}r}{\mathrm{d}t}$ when $A = 1900$ m^2.

$A = \pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$.

When $A = 1900$: $r = \sqrt{\frac{1900}{\pi}}$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} \div \frac{\mathrm{d}A}{\mathrm{d}r} = \frac{250}{2\pi r} = \frac{250}{2\pi\sqrt{1900/\pi}} = \frac{250}{2\sqrt{1900\pi}}$$
$$= \frac{125}{\sqrt{1900\pi}} \approx \frac{125}{77.26\ldots} \approx 1.6 \text{ m}\,\text{hr}^{-1}$$

$\frac{\mathrm{d}r}{\mathrm{d}t} \approx 1.6 \text{ m}\,\text{hr}^{-1}$ (2 s.f.)

Question 4 (Jan 2011, Q3)

Worked Solution

Spherical balloon: $S = 4\pi r^2$, $\dfrac{\mathrm{d}r}{\mathrm{d}t} = 12 \text{ cm hr}^{-1}$. Find $\dfrac{\mathrm{d}S}{\mathrm{d}t}$ when $r = 150$ cm.

$\dfrac{\mathrm{d}S}{\mathrm{d}r} = 8\pi r.$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r \times 12 = 96\pi r$$

When $r = 150$:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 96\pi \times 150 = 14400\pi \approx 45239 \text{ cm}^2\text{hr}^{-1}$$

$\dfrac{\mathrm{d}S}{\mathrm{d}t} = 14400\pi \approx 45000 \text{ cm}^2\,\text{hr}^{-1}$ (2 s.f.)

Question 5 (Jun 2012, Q6)

Worked Solution

$V = (3h^2 + 4)^{3/2} - 8$.

(i) Find $\frac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 0.6$:

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{3}{2}(3h^2+4)^{1/2} \cdot 6h = 9h(3h^2+4)^{1/2}$$

When $h = 0.6$: $3(0.36) + 4 = 1.08 + 4 = 5.08$.

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 9(0.6)(5.08)^{1/2} = 5.4\sqrt{5.08} \approx 5.4 \times 2.2539\ldots \approx 12.17$$

$\frac{\mathrm{d}V}{\mathrm{d}h} \approx 12.17$ when $h = 0.6$ (2 d.p.)

(ii) Rate of decrease of volume when $h = 0.6$ m, given depth decreasing at 0.015 m hr^{-1}:

$\frac{\mathrm{d}h}{\mathrm{d}t} = -0.015 \text{ m hr}^{-1}$.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t} = 12.17 \times (-0.015) \approx -0.18 \text{ m}^3\text{hr}^{-1}$$

Rate of decrease of volume $\approx 0.18 \text{ m}^3 \text{ hr}^{-1}$

Question 6 (Jun 2013, Q3)

Worked Solution

Cone with half-angle $\alpha = \tan^{-1}(\frac{1}{2})$, so radius $= \frac{x}{2}$ when depth is x. Liquid added at $14\ \text{cm}^3\,\text{min}^{-1}$.

(i) Show $V = \dfrac{1}{12}\pi x^3$**:**

Since $\tan\alpha = \frac{1}{2}$, radius $r = \frac{x}{2}$.

$$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi\left(\frac{x}{2}\right)^2 x = \frac{1}{3}\pi \cdot \frac{x^2}{4} \cdot x = \frac{\pi x^3}{12} \quad \checkmark$$

(ii) Rate of increase of depth when $x = 8$ **cm:**

$\dfrac{\mathrm{d}V}{\mathrm{d}x} = \dfrac{\pi x^2}{4}$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}x} = \frac{14}{\pi x^2/4} = \frac{56}{\pi x^2}$$

When $x = 8$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{56}{64\pi} = \frac{7}{8\pi} \approx 0.28\ \text{cm}\,\text{min}^{-1}$$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} \approx 0.28\ \text{cm}\,\text{min}^{-1}$ (2 d.p.)

Question 7 (Jun 2015, Q3)

Worked Solution

$V = 3(2+\sqrt{h})^6 - 192$, $\quad \dfrac{dV}{dt} = 150 \text{ m}^3\,\text{hr}^{-1}$. Find $\dfrac{dh}{dt}$ when $h = 1.4$.

$$\frac{dV}{dh} = 3 \cdot 6(2+\sqrt{h})^5 \cdot \frac{1}{2\sqrt{h}} = \frac{9(2+\sqrt{h})^5}{\sqrt{h}}$$

When $h = 1.4$: $\sqrt{1.4} = 1.1832\ldots$, so $2+\sqrt{h} = 3.1832\ldots$

$$\frac{dV}{dh} = \frac{9(3.1832)^5}{1.1832} = \frac{9 \times 327.07\ldots}{1.1832} \approx \frac{2943.6}{1.1832} \approx 2488 \quad (\text{or } 9h^{-1/2}(2+\sqrt{h})^5)$$

More precisely: $9 \times (1.1832)^{-1} \times (3.1832)^5$. Let $u = 3.1832$: $u^5 = 328.33\ldots$

$$\frac{dV}{dh} = \frac{9 \times 328.33}{1.1832} \approx 2498$$

$$\frac{dh}{dt} = \frac{150}{dV/dh} \approx \frac{150}{2498} \approx 0.060 \text{ m}\,\text{hr}^{-1}$$

$\dfrac{dh}{dt} \approx 0.060 \text{ m}\,\text{hr}^{-1}$ (2 s.f.)

Question 8 (Jun 2018, Q6a)

Worked Solution

$V = 2(5+2x)^3 - 250, \quad \dfrac{\mathrm{d}V}{\mathrm{d}t} = 15\ \mathrm{m}^3\,\mathrm{min}^{-1}$. Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 1.6$.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 2 \times 3(5+2x)^2 \times 2 = 12(5+2x)^2$$

When $x = 1.6$: $5 + 2(1.6) = 8.2$.

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 12(8.2)^2 = 12 \times 67.24 = 806.88$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{15}{806.88} \approx 0.0186\ \mathrm{m}\,\mathrm{min}^{-1}$$

$\dfrac{\mathrm{d}x}{\mathrm{d}t} \approx 0.019\ \mathrm{m}\,\mathrm{min}^{-1}$ (2 s.f.)

End of Worked Solutions