Question 1

Worked Solution

Area of a circle: $A = \pi r^2$, so $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$.

Given $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 1.5\ \mathrm{cm^2 s^{-1}}$.

When $A = 2$: $2 = \pi r^2 \Rightarrow r = \sqrt{\dfrac{2}{\pi}}$.

By the chain rule:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \implies 1.5 = 2\pi r \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \implies \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1.5}{2\pi\sqrt{2/\pi}} = \frac{1.5}{2\sqrt{2\pi}}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx 0.299\ \mathrm{cm\,s^{-1}}\ (3\text{ s.f.})$$

Question 2

Worked Solution

$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 3 \text{ cm}^3\text{s}^{-1}$.

(a) Find $\dfrac{\mathrm{d}r}{\mathrm{d}t}$ when $r = 4$ cm:

$\dfrac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}r} = \frac{3}{4\pi r^2}$$

When $r = 4$:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{4\pi(16)} = \frac{3}{64\pi}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{64\pi} \approx 0.0149 \text{ cm s}^{-1} \text{ (3 s.f.)}$$

(b) Rate of increase of surface area when $r = 4$ cm:

$\dfrac{\mathrm{d}S}{\mathrm{d}r} = 8\pi r$.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = 8\pi r \times \frac{3}{4\pi r^2} = \frac{6}{r}$$

When $r = 4$:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{6}{4} = 1.5$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 1.5 \text{ cm}^2\text{s}^{-1}$$

Question 3

Worked Solution

$V = 4\pi h(h+4) = 4\pi h^2 + 16\pi h, \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 80\pi \text{ cm}^3\text{s}^{-1}.$

Differentiate V with respect to h:

$$\frac{\mathrm{d}V}{\mathrm{d}h} = 8\pi h + 16\pi$$

By the chain rule:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h} = \frac{80\pi}{8\pi h + 16\pi}$$

When $h = 6$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{80\pi}{48\pi + 16\pi} = \frac{80\pi}{64\pi} = \frac{80}{64} = \frac{5}{4}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 1.25 \text{ cm s}^{-1}$$

Question 4

Worked Solution

Cylinder: radius x cm, length $5x$ cm. Cross-sectional area $A = \pi x^2$, volume $V = \pi x^2 \cdot 5x = 5\pi x^3$.

Given $\dfrac{\mathrm{d}A}{\mathrm{d}t} = 0.032\ \mathrm{cm}^2\mathrm{s}^{-1}$.

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 2$ cm:

$\dfrac{\mathrm{d}A}{\mathrm{d}x} = 2\pi x$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t} \div \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{0.032}{2\pi x} = \frac{0.016}{\pi x}$$

When $x = 2$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{0.016}{2\pi} = \frac{0.008}{\pi}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{0.016}{2\pi} \approx 0.00255\ \mathrm{cm\,s}^{-1}\ (3\text{ s.f.})$$

(b) Rate of increase of volume when $x = 2$:

$\dfrac{\mathrm{d}V}{\mathrm{d}x} = 15\pi x^2$.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = 15\pi x^2 \times \frac{0.016}{\pi x} = 15x \times 0.016 = 0.24x$$

When $x = 2$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.24 \times 2 = 0.48$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0.48\ \mathrm{cm}^3\mathrm{s}^{-1}$$

Question 5

Worked Solution

$V = \frac{1}{12}\pi h^2(3 - 4h), \quad 0 \le h \le 0.25, \quad \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{800}\ \mathrm{m^3 s^{-1}}.$

(a) Find $\frac{\mathrm{d}V}{\mathrm{d}h}$ when $h = 0.1$:

Expand: $V = \frac{\pi}{12}(3h^2 - 4h^3)$.

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{12}(6h - 12h^2) = \frac{\pi}{2}h - \pi h^2$$

When $h = 0.1$:

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{2}(0.1) - \pi(0.01) = 0.05\pi - 0.01\pi = 0.04\pi = \frac{\pi}{25}$$

$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{25}$ at $h = 0.1$

(b) Rate of change of h when $h = 0.1$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h} = \frac{\pi}{800} \times \frac{25}{\pi} = \frac{25}{800} = \frac{1}{32}$$

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{32} \approx 0.031\ \mathrm{m\,s^{-1}}$

Question 6

Worked Solution

Inverted cone: full height 24 cm, full radius 16 cm. Water depth h cm, surface radius r cm.

(a) Show $V = \dfrac{4\pi h^3}{27}$:

By similar triangles: $\dfrac{r}{h} = \dfrac{16}{24} = \dfrac{2}{3}$, so $r = \dfrac{2h}{3}$.

Substituting into $V = \frac{1}{3}\pi r^2 h$:

$$V = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h = \frac{1}{3}\pi \cdot \frac{4h^2}{9} \cdot h = \frac{4\pi h^3}{27} \quad \checkmark$$

(b) Find $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ in terms of π when $h = 12$, given $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 8$ cm^{3}s^{-1}:

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{12\pi h^2}{27} = \frac{4\pi h^2}{9}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h} = 8 \times \frac{9}{4\pi h^2} = \frac{18}{\pi h^2}$$

When $h = 12$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{18}{\pi \times 144} = \frac{18}{144\pi} = \frac{1}{8\pi}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{8\pi}\ \mathrm{cm\,s^{-1}}$$

End of Worked Solutions