Question 1 (Jun 2006, Q8)

Worked Solution

(i) Express $5\cos x + 12\sin x$ in the form $R\cos(x-\alpha)$, $R > 0$, $0° < \alpha < 90°$:

$R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$.

Matching: $R\cos\alpha = 5$, $R\sin\alpha = 12$.

$$R = \sqrt{5^2+12^2} = \sqrt{169} = 13$$
$$\tan\alpha = \tfrac{12}{5} \implies \alpha = 67.38\ldots° \approx 67.4°$$

$5\cos x + 12\sin x = 13\cos(x - 67.4°)$

(ii) Pair of transformations from $y = \cos x$ to $y = 5\cos x + 12\sin x$:

$y = 13\cos(x-67.4°)$: translation of 67.4° in positive x-direction, then stretch in y-direction by factor 13 (in either order).

Translation by 67.4° parallel to x-axis (positive direction); stretch parallel to y-axis, scale factor 13.

(iii) Solve $5\cos x + 12\sin x = 2$ for $0° < x < 360°$:

$$13\cos(x-67.4°) = 2 \implies \cos(x-67.4°) = \frac{2}{13}$$
$$x - 67.4° = \cos^{-1}\left(\tfrac{2}{13}\right) = 81.15\ldots°$$

First solution: $x = 81.15° + 67.4° = 148.5°$

Second solution: $x - 67.4° = -81.15° \Rightarrow x = -13.8°$ (not in range), so use $x - 67.4° = 360° - 81.15° = 278.85° \Rightarrow x = 346.2°$.

$x = 148.5°$ and $x = 346.2°$

Question 2 (Jan 2011, Q4)

Worked Solution

(i) Express $24\sin\theta + 7\cos\theta$ in the form $R\sin(\theta+\alpha)$, $R > 0$, $0° < \alpha < 90°$:

$R\sin(\theta+\alpha) = R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$.

Matching: $R\cos\alpha = 24$, $R\sin\alpha = 7$.

$$R = \sqrt{24^2 + 7^2} = \sqrt{576+49} = \sqrt{625} = 25$$
$$\tan\alpha = \tfrac{7}{24} \implies \alpha = 16.26\ldots° \approx 16.3°$$

$24\sin\theta + 7\cos\theta = 25\sin(\theta + 16.3°)$

(ii) Solve $24\sin\theta + 7\cos\theta = 12$ for $0° < \theta < 360°$:

$$25\sin(\theta + 16.26°) = 12 \implies \sin(\theta+16.26°) = 0.48$$
$$\theta + 16.26° = \arcsin(0.48) = 28.69\ldots° \quad \text{or} \quad 180° - 28.69° = 151.31°$$

First: $\theta = 28.69° - 16.26° = 12.4°$

Second: $\theta = 151.31° - 16.26° = 135.1°$ (or $135.0°$)

$\theta = 12.4°$ and $\theta = 135.1°$

Question 3 (Jun 2012, Q8)

Worked Solution

(i) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$, $R > 0$, $0° < \alpha < 90°$:

Matching: $R\cos\alpha = 3$, $R\sin\alpha = 4$.

$$R = \sqrt{9+16} = 5, \quad \tan\alpha = \tfrac{4}{3} \implies \alpha = 53.13\ldots° \approx 53.1°$$

$3\sin\theta + 4\cos\theta = 5\sin(\theta + 53.1°)$

(ii)(a) Solve $3\sin\theta + 4\cos\theta + 1 = 0$ for $-180° < \theta < 180°$:

$$5\sin(\theta + 53.13°) = -1 \implies \sin(\theta + 53.13°) = -0.2$$
$$\theta + 53.13° = \arcsin(-0.2) = -11.54° \quad \text{or} \quad 180° - (-11.54°) = 191.54°$$

Range of $\theta + 53.13°$: $(-180° + 53.13°, 180° + 53.13°) = (-126.87°, 233.13°)$.

$\theta + 53.13° = -11.54° \Rightarrow \theta = -64.7°$

$\theta + 53.13° = 191.54° \Rightarrow \theta = 138°$

$\theta = -64.7°$ and $\theta = 138°$

(ii)(b) Find positive constants k and c such that $-37 \le k(3\sin\theta + 4\cos\theta) + c \le 43$:

Since $-5 \le 3\sin\theta + 4\cos\theta \le 5$, we have $-5k \le k(3\sin\theta + 4\cos\theta) \le 5k$.

So $c - 5k \le k(3\sin\theta + 4\cos\theta) + c \le c + 5k$.

Setting $c - 5k = -37$ and $c + 5k = 43$:

Adding: $2c = 6 \Rightarrow c = 3$.

Subtracting: $10k = 80 \Rightarrow k = 8$.

$k = 8$, $c = 3$

Question 4 (Jun 2013, Q8)

Worked Solution

(i) Express $4\cos\theta - 2\sin\theta$ in the form $R\cos(\theta+\alpha)$, $R > 0$, $0° < \alpha < 90°$:

$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$.

Matching: $R\cos\alpha = 4$, $R\sin\alpha = 2$.

$$R = \sqrt{16+4} = \sqrt{20} \approx 4.47, \quad \tan\alpha = \tfrac{2}{4} = 0.5 \implies \alpha = 26.57\ldots° \approx 26.6°$$

$4\cos\theta - 2\sin\theta = \sqrt{20}\cos(\theta + 26.6°)$

(ii)(a) Solve $4\cos\theta - 2\sin\theta = 3$ for $0° < \theta < 360°$:

$$\sqrt{20}\cos(\theta + 26.57°) = 3 \implies \cos(\theta+26.57°) = \frac{3}{\sqrt{20}} = 0.6708\ldots$$
$$\theta + 26.57° = 47.87\ldots° \quad \text{or} \quad 360° - 47.87° = 312.13°$$

First: $\theta = 47.87° - 26.57° = 21.3°$

Second: $\theta = 312.13° - 26.57° = 285.6°$ (or 285.6°)

$\theta = 21.3°$ and $\theta = 285.6°$

(ii)(b) Greatest and least values of $25-(4\cos\theta-2\sin\theta)^2$, and smallest positive θ for each:

$(4\cos\theta - 2\sin\theta)^2 = 20\cos^2(\theta + 26.57°)$, which ranges from 0 to 20.

Greatest value of $25 - 20\cos^2(\theta + 26.57°)$: when $\cos^2(\theta+26.57°) = 0$:

$$\theta + 26.57° = 90° \implies \theta = 63.4°$$

Greatest value $= 25$.

Least value of expression: when $\cos^2(\theta + 26.57°) = 1$:

$$\theta + 26.57° = 0° \text{ (not in range) or } 360° \implies \theta = 333.4°$$

But smallest positive θ: $\theta+26.57° = 360° \Rightarrow \theta = 333.4°$. However $\cos(\theta+26.57°) = -1$ at $\theta + 26.57° = 180° \Rightarrow \theta = 153.4°$ also gives $\cos^2 = 1$. Smallest positive is $\theta = 153.4°$.

Least value $= 25 - 20 = 5$.

Greatest value: 25, at $\theta = 63.4°$ (smallest positive)
Least value: 5, at $\theta = 153.4°$ (smallest positive)

Question 5 (Jun 2014, Q9)

Worked Solution

(i) Express $5\cos(\theta-60°)+3\cos\theta$ in the form $R\sin(\theta+\alpha)$, $R>0$, $0°<\alpha<90°$:

Expand $5\cos(\theta-60°)$:

$5(\cos\theta\cos 60°+\sin\theta\sin 60°)+3\cos\theta = 5(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta)+3\cos\theta = \frac{5}{2}\cos\theta+\frac{5\sqrt{3}}{2}\sin\theta+3\cos\theta = \frac{11}{2}\cos$

Write as $R\sin(\theta+\alpha) = R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$:

$R\cos\alpha = \frac{5\sqrt{3}}{2}$, $R\sin\alpha = \frac{11}{2}$.

$$R = \sqrt{\left(\frac{5\sqrt{3}}{2}\right)^2+\left(\frac{11}{2}\right)^2} = \sqrt{\frac{75}{4}+\frac{121}{4}} = \sqrt{\frac{196}{4}} = 7$$

$$\tan\alpha = \frac{11/2}{5\sqrt{3}/2} = \frac{11}{5\sqrt{3}} \implies \alpha = 51.8°$$

$5\cos(\theta-60°)+3\cos\theta = 7\sin(\theta+51.8°)$

(ii)(a) Transformations from $y=\sin\theta$ to $y=5\cos(\theta-60°)+3\cos\theta$:

$y=7\sin(\theta+51.8°)$: translation by $51.8°$ in the negative θ-direction (i.e. $\begin{pmatrix}-51.8\\0\end{pmatrix}$), then stretch parallel to y-axis by factor 7 (in either order).

Stretch parallel to y-axis with factor $\frac{1}{7}$ and translation by $\begin{pmatrix}-51.8\\0\end{pmatrix}$ (to go from $y=7\sin(\theta+51.8°)$ to $y=\sin\theta$); in reverse: translation $\begin{pmatrix}-51.8\\0\end{pmatrix}$ and stretch factor 7 parallel to y-axis.

(ii)(b) Smallest positive β satisfying $5\cos(\frac{1}{3}\beta-40°)+3\cos(\frac{1}{3}\beta+20°)=3$:

Let $\theta=\frac{1}{3}\beta+20°$, so $\theta-60°=\frac{1}{3}\beta-40°$.

The equation becomes $5\cos(\theta-60°)+3\cos\theta=3$, i.e. $7\sin(\theta+51.8°)=3$:

$$\sin(\theta+51.8°)=\frac{3}{7}$$

$$\theta+51.8° = \arcsin(\tfrac{3}{7}) = 25.38° \quad\text{or}\quad 154.62°$$

Since $\beta>0$, $\theta=\frac{1}{3}\beta+20°>20°$, so $\theta+51.8°>71.8°$.

Use $\theta+51.8°=154.62° \Rightarrow \theta=102.82° \Rightarrow \frac{1}{3}\beta=82.82° \Rightarrow \beta=248.5°$.

(The $25.38°$ solution gives $\theta=-26.4°$, meaning $\beta<0$; next cycle: $\theta+51.8° = 360°+25.38°=385.38°$ gives larger β.)

$\beta=248°$ (or $249°$; to 3 s.f.: $248.5°$)

End of Worked Solutions