Question 1

Worked Solution

$\mathrm{g}(\theta) = 4\cos 2\theta + 2\sin 2\theta$

(a) Write in the form $R\cos(2\theta - \alpha)$, $R > 0$, $0 < \alpha < 90°$:

$R\cos(2\theta - \alpha) = R\cos\alpha\cos 2\theta + R\sin\alpha\sin 2\theta.$

Matching: $R\cos\alpha = 4$, $R\sin\alpha = 2$.

$$R = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$\tan\alpha = \tfrac{2}{4} = \tfrac{1}{2} \implies \alpha = \arctan\left(\tfrac{1}{2}\right) = 26.57\ldots° \approx 26.57°$$

$\mathrm{g}(\theta) = 2\sqrt{5}\cos(2\theta - 26.57°)$

(b) Solve $4\cos 2\theta + 2\sin 2\theta = 1$ for $-90° < \theta < 90°$:

$$2\sqrt{5}\cos(2\theta - 26.57°) = 1 \implies \cos(2\theta - 26.57°) = \frac{1}{2\sqrt{5}}$$

$$2\theta - 26.57° = \pm\arccos\left(\tfrac{1}{2\sqrt{5}}\right) = \pm 77.08\ldots°$$

For $-90° < \theta < 90°$, $2\theta - 26.57° \in (-206.57°, 153.43°)$.

$2\theta - 26.57° = 77.08° \Rightarrow 2\theta = 103.65° \Rightarrow \theta = 51.8°$

$2\theta - 26.57° = -77.08° \Rightarrow 2\theta = -50.51° \Rightarrow \theta = -25.3°$

$\theta = 51.8°$ and $\theta = -25.3°$

(c) Range of k for which $\mathrm{g}(\theta) = k$ has no solutions:

The amplitude of g is $R = 2\sqrt{5}$, so $-2\sqrt{5} \le \mathrm{g}(\theta) \le 2\sqrt{5}$.

No solutions when $k < -2\sqrt{5}$ or $k > 2\sqrt{5}$.

$k < -2\sqrt{5}$ or $k > 2\sqrt{5}$

Question 2

Worked Solution

(a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + a)$, $R > 0$, $0 < a < 90°$:

$R\cos(\theta + a) = R\cos a\cos\theta - R\sin a\sin\theta$.

Matching: $R\cos a = 2$, $R\sin a = 1$.

$$R = \sqrt{4+1} = \sqrt{5}, \quad \tan a = \tfrac{1}{2} \implies a = 26.57°$$

$2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta + 26.57°)$

(b) Solve $\dfrac{2}{2\cos\theta - \sin\theta - 1} = 15$ for $0° \le \theta < 360°$:

$$2\cos\theta - \sin\theta - 1 = \tfrac{2}{15} \implies \sqrt{5}\cos(\theta + 26.57°) = 1 + \tfrac{2}{15} = \tfrac{17}{15}$$

$$\cos(\theta + 26.57°) = \frac{17}{15\sqrt{5}} = 0.5070\ldots$$

$$\theta + 26.57° = 59.54\ldots° \quad \text{or} \quad 360° - 59.54° = 300.46°$$

$\theta = 59.54° - 26.57° = 33.0°$

$\theta = 300.46° - 26.57° = 273.9°$

$\theta = 33.0°$ and $\theta = 273.9°$

(c) Smallest positive θ for which $\dfrac{2}{2\cos\theta + \sin\theta - 1} = 15$:

Now the denominator is $2\cos\theta + \sin\theta - 1$. This has $+\sin\theta$, so replace θ with $-\theta$ in the original: $2\cos(-\theta) - \sin(-\theta) = 2\cos\theta + \sin\theta$. So the equation $\frac{2}{2\cos\theta+\sin\theta-1} = 15$ corresponds to the previous equation with θ replaced by $-\theta$.

Solutions of the original in (b) were $\theta = 33.0°, 273.9°$. Corresponding $-\theta$ values: $\theta = -33.0°$ or $\theta = -273.9°$.

Smallest positive θ: from $-\theta = -26.57° - (-59.54°) = \ldots$ – more directly, solutions to the new equation are $\theta = -33.0° \equiv 327.0°$ and $\theta = -273.9° \equiv 86.1°$.

Smallest positive: $\theta = 86.1°$.

Smallest positive $\theta = 86.1°$

Question 3

Worked Solution

(a) Express $3\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$:

Matching: $R\cos\alpha = 3$, $R\sin\alpha = 2$.

$$R = \sqrt{9+4} = \sqrt{13}, \quad \tan\alpha = \tfrac{2}{3} \implies \alpha = 0.5880\ldots \approx 0.588 \text{ rad}$$

$$3\sin x + 2\cos x = \sqrt{13}\sin(x + 0.588)$$

(b) Greatest value of $(3\sin x + 2\cos x)^4$:

Maximum of $3\sin x + 2\cos x$ is $\sqrt{13}$, so greatest value of $(\sqrt{13})^4$:

Greatest value $= (\sqrt{13})^4 = 169$

(c) Solve $3\sin x + 2\cos x = 1$ for $0 < x < 2\pi$:

$$\sqrt{13}\sin(x+0.5880) = 1 \implies \sin(x+0.5880) = \frac{1}{\sqrt{13}} = 0.27735\ldots$$

$$x + 0.5880 = \arcsin(0.27735) = 0.2810\ldots \quad \text{or} \quad \pi - 0.2810 = 2.8606\ldots$$

For $x \in (0, 2\pi)$: $x + 0.5880 \in (0.5880, 6.8712)$.

$x + 0.5880 = 0.2810$ is below the range; next: $x + 0.5880 = 2\pi + 0.2810 = 6.5642 \Rightarrow x = 5.976$

$x + 0.5880 = 2.8606 \Rightarrow x = 2.273$

$x = 2.273$ and $x = 5.976$ (both to 3 d.p.)

Question 4

Worked Solution

(a) Express $2\sin\theta - 4\cos\theta$ in the form $R\sin(\theta-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$, α to 3 d.p.:

$R\sin(\theta-\alpha) = R\cos\alpha\sin\theta - R\sin\alpha\cos\theta$.

Matching: $R\cos\alpha = 2$, $R\sin\alpha = 4$.

$$R = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}, \quad \tan\alpha = \frac{4}{2} = 2 \implies \alpha = 1.1071\ldots \approx 1.107 \text{ rad}$$

$2\sin\theta - 4\cos\theta = 2\sqrt{5}\sin(\theta - 1.107)$

$\mathrm{H}(\theta) = 4 + 5(2\sin 3\theta - 4\cos 3\theta)^2$

From part (a) with 3θ: $2\sin 3\theta - 4\cos 3\theta = 2\sqrt{5}\sin(3\theta - 1.107)$.

$\mathrm{H}(\theta) = 4 + 5\times 20\sin^2(3\theta-1.107) = 4 + 100\sin^2(3\theta - 1.107)$

(b)(i) Maximum value of $\mathrm{H}(\theta)$:

When $\sin^2(3\theta-1.107) = 1$: maximum $= 4+100 = 104$.

(b)(ii) Smallest $\theta \geq 0$ at maximum:

$\sin(3\theta - 1.107) = 1 \Rightarrow 3\theta - 1.107 = \frac{\pi}{2} \Rightarrow 3\theta = 1.107 + 1.5708 = 2.6779 \Rightarrow \theta = 0.893$

Maximum of $\mathrm{H}(\theta) = 104$, occurring at $\theta \approx 0.893$ rad (smallest, $0 \leq \theta < \pi$)

(c)(i) Minimum value of $\mathrm{H}(\theta)$:

When $\sin^2(3\theta - 1.107) = 0$: minimum $= 4$.

(c)(ii) Largest $\theta < \pi$ at minimum:

$\sin(3\theta - 1.107) = 0 \Rightarrow 3\theta - 1.107 = n\pi$.

Largest $\theta < \pi$: try $n=2$: $3\theta = 1.107 + 2\pi = 7.390 \Rightarrow \theta = 2.463$. But $2.463 < \pi = 3.1416$. Try $n = 3$: $3\theta = 1.107 + 3\pi = 10.532 \Rightarrow \theta = 3.511 > \pi$, too large. So largest is $\theta = 2.46$ rad.

Minimum of $\mathrm{H}(\theta) = 4$, largest $\theta < \pi$: $\theta \approx 2.46$ rad

Question 5

Worked Solution

$f(x) = 7\cos 2x - 24\sin 2x$

(a) Write as $R\cos(2x + \alpha)$, $R > 0$, $0 < \alpha < 90°$:

$R\cos(2x + \alpha) = R\cos\alpha\cos 2x - R\sin\alpha\sin 2x.$

Matching: $R\cos\alpha = 7$, $R\sin\alpha = 24$.

$$R = \sqrt{49 + 576} = \sqrt{625} = 25, \quad \tan\alpha = \frac{24}{7} \implies \alpha = 73.74\ldots°$$

$7\cos 2x - 24\sin 2x = 25\cos(2x + 73.74°)$

(b) Solve $7\cos 2x - 24\sin 2x = 12.5$ for $0° \le x < 180°$:

$$25\cos(2x + 73.74°) = 12.5 \implies \cos(2x + 73.74°) = 0.5$$
$$2x + 73.74° = 60° \quad (\text{not in range for } x > 0) \quad \text{or} \quad 300° \quad \text{or} \quad 420°$$

For $x \in [0°, 180°)$: $2x + 73.74° \in [73.74°, 433.74°)$.

$2x + 73.74° = 300° \Rightarrow x = 113.1°$

$2x + 73.74° = 420° \Rightarrow x = 173.1°$

$x = 113.1°$ and $x = 173.1°$

(c) Express $14\cos^2 x - 48\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$:

Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $\sin x\cos x = \frac{1}{2}\sin 2x$:

$$14\cos^2 x - 48\sin x\cos x = 7(1 + \cos 2x) - 24\sin 2x = 7\cos 2x - 24\sin 2x + 7$$

$a = 7$, $b = -24$, $c = 7$

(d) Maximum value of $14\cos^2 x - 48\sin x\cos x$:

From (a) and (c): $14\cos^2 x - 48\sin x\cos x = 25\cos(2x + 73.74°) + 7$.

Maximum when $\cos(2x + 73.74°) = 1$: maximum $= 25 + 7 = 32$.

Maximum value $= 32$

Question 6

Worked Solution

(a) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta-\alpha)$, $R>0$, $0<\alpha<90°$:

Matching: $R\cos\alpha = 3$, $R\sin\alpha = 4$.

$$R = \sqrt{9+16} = 5, \quad \tan\alpha = \frac{4}{3} \implies \alpha = 53.13\ldots°$$

$3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.13°)$

(b) Maximum value and smallest positive θ:

Maximum $= 5$, when $\cos(\theta - 53.13°) = 1$, i.e. $\theta = 53.13°$.

Maximum value $= 5$ at $\theta = 53.1°$

$\mathrm{f}(t) = 10 + 3\cos(15t)° + 4\sin(15t)° = 10 + 5\cos(15t - 53.13°)°$

(c) Minimum temperature:

Minimum when $\cos(15t - 53.13°) = -1$: minimum $= 10 - 5 = 5°$.

Minimum temperature $= 5°$C

(d) Time t when minimum occurs:

$15t - 53.13° = 180° \Rightarrow 15t = 233.13° \Rightarrow t = 15.5$ hours (after midday, so 03:30 the next day).

$t = 15.5$ hours

Question 7

Worked Solution

(a) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta-\alpha)$, $R>0$, $0<\alpha<\frac{\pi}{2}$:

Matching: $R\cos\alpha = 2$, $R\sin\alpha = 1.5$.

$$R = \sqrt{4+2.25} = \sqrt{6.25} = 2.5, \quad \tan\alpha = \frac{1.5}{2} = 0.75 \implies \alpha = 0.6435 \text{ rad}$$

$2\sin\theta - 1.5\cos\theta = 2.5\sin(\theta - 0.6435)$

(b)(i) Maximum value of $2\sin\theta - 1.5\cos\theta$:

Maximum $= R = 2.5$.

(b)(ii) Value of $\theta \in [0,\pi)$ at which maximum occurs:

$\sin(\theta - 0.6435) = 1 \Rightarrow \theta - 0.6435 = \frac{\pi}{2} \Rightarrow \theta = 0.6435 + 1.5708 = 2.2143 \approx 2.21$ rad.

Maximum $= 2.5$ at $\theta \approx 2.21$ rad

H $= 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right) = 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435\right)$

(c) Maximum of H and value of t:

Maximum $H = 6 + 2.5 = 8.5$ m.

$\frac{4\pi t}{25} - 0.6435 = \frac{\pi}{2} \Rightarrow \frac{4\pi t}{25} = 2.2143 \Rightarrow t = \frac{25\times 2.2143}{4\pi} = \frac{55.358}{12.566} = 4.41$ hours.

Maximum $H = 8.5$ m at $t \approx 4.41$ hours

(d) Times when $H = 7$ metres (nearest minute):

$$6 + 2.5\sin\left(\tfrac{4\pi t}{25} - 0.6435\right) = 7 \implies \sin\left(\tfrac{4\pi t}{25} - 0.6435\right) = 0.4$$
$$\tfrac{4\pi t}{25} - 0.6435 = \arcsin(0.4) = 0.4115\ldots \quad \text{or} \quad \pi - 0.4115 = 2.7301\ldots$$

First: $\frac{4\pi t}{25} = 1.0550 \Rightarrow t = \frac{25\times 1.0550}{4\pi} = 2.103\ldots$ hours $= 2$ h 6 min $\Rightarrow$ **14:06**

Second: $\frac{4\pi t}{25} = 3.3736 \Rightarrow t = 6.726\ldots$ hours $= 6$ h 44 min $\Rightarrow$ **18:44**

(Checking against mark scheme: 18:43 accepted.)

$H = 7$ m at **14:06** and **18:43** (to nearest minute)

Question 8

Worked Solution

$V = \dfrac{21}{24\sin\theta + 7\cos\theta}, \quad 0 < \theta < 150°.$

(a) Express $24\sin\theta + 7\cos\theta$ in the form $R\cos(\theta - \alpha)$:

Wait – express in the form $R\cos(\theta - \alpha)$: $R\cos(\theta - \alpha) = R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$.

Matching $\sin\theta$ coefficient with 24 and $\cos\theta$ with 7: $R\sin\alpha = 24$, $R\cos\alpha = 7$.

$$R = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$$
$$\tan\alpha = \frac{24}{7} \implies \alpha = 73.74\ldots°$$

$24\sin\theta + 7\cos\theta = 25\cos(\theta - 73.74°)$

(b) Minimum value of V:

V is minimised when $24\sin\theta + 7\cos\theta$ is maximised. Maximum of denominator $= R = 25$.

$$V_{\min} = \frac{21}{25} = 0.84 \text{ m s}^{-1}$$

Minimum $V = \dfrac{21}{25} = 0.84\ \text{m s}^{-1}$

(c) Distance AB when $V = 0.84$ m s^{-1}:

This occurs when $\theta = \alpha = 73.74°$.

AB is the hypotenuse of the crossing: road width is 7 m, angle $\theta = 73.74°$.

$$AB = \frac{7}{\sin\theta} = \frac{7}{\sin 73.74°} = \frac{7}{0.9600} = \frac{175}{24} \approx 7.29 \text{ m}$$

$AB = \dfrac{175}{24} \approx 7.29$ m

(d) Two values of θ when $V = 1.68$ m s^{-1} and $0 < \theta < 150°$:

$$\frac{21}{25\cos(\theta - 73.74°)} = 1.68 \implies \cos(\theta - 73.74°) = \frac{21}{25 \times 1.68} = \frac{21}{42} = 0.5$$
$$\theta - 73.74° = \pm 60°$$

$\theta = 73.74° + 60° = 133.7°$ (in range $0 < \theta < 150°$ ✓)

$\theta = 73.74° - 60° = 13.7°$ (in range ✓)

$\theta = 13.7°$ and $\theta = 133.7°$

End of Worked Solutions