Question 1 (Jan 2009, Q9)

Worked Solution

(i) Prove $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$:

Write $\cos 3\theta = \cos(2\theta + \theta)$ and expand:

$$\cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$

Substitute $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$:

$$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\cdot\sin\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

Replace $\sin^2\theta = 1 - \cos^2\theta$:

$$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ✓

(ii) Hence prove $\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$:

Use $\cos 6\theta = 2\cos^2 3\theta - 1$, substituting the result from (i):

$$\cos 6\theta = 2(4\cos^3\theta - 3\cos\theta)^2 - 1$$

Expand $(4\cos^3\theta - 3\cos\theta)^2 = 16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta$:

$$\cos 6\theta = 2(16\cos^6\theta - 24\cos^4\theta + 9\cos^2\theta) - 1$$

$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$ ✓

(iii) Show the only solutions of $1 + \cos 6\theta = 18\cos^2\theta$ are odd multiples of $90°$:

Substitute the result from (ii):

$$1 + 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = 18\cos^2\theta$$

$$32\cos^6\theta - 48\cos^4\theta = 0$$

$$16\cos^4\theta(2\cos^2\theta - 3) = 0$$

Let $c = \cos\theta$: either $c^4 = 0$ or $2c^2 = 3$.

$2c^2 = 3 \Rightarrow c^2 = \frac{3}{2} > 1$: no real solutions.

$c = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°, 270°, \ldots$ i.e. odd multiples of $90°$.

The only solutions are odd multiples of $90°$. ✓

Question 2 (Jan 2010, Q9)

Worked Solution

Let $\tan 10° = p$.

(i) $\tan 55°$ in terms of p:

Write $55° = 45° + 10°$ and use the addition formula:

$$\tan 55° = \tan(45° + 10°) = \frac{\tan 45° + \tan 10°}{1 - \tan 45° \tan 10°} = \frac{1+p}{1-p}$$

$$\tan 55° = \frac{1+p}{1-p}$$

(ii) $\tan 5°$ in terms of p:

Use the double angle formula $\tan 10° = \tan(2 \times 5°)$:

$$p = \frac{2t}{1-t^2} \quad \text{where } t = \tan 5°$$

$$p(1-t^2) = 2t \implies pt^2 + 2t - p = 0$$

$$t = \frac{-2 \pm \sqrt{4+4p^2}}{2p} = \frac{-1 \pm \sqrt{1+p^2}}{p}$$

Since $\tan 5° > 0$, take the positive root:

$$\tan 5° = \frac{-1+\sqrt{1+p^2}}{p}$$

(iii) $\tan\theta$ where $3\sin(\theta + 10°) = 7\cos(\theta - 10°)$:

Expand both sides:

$$3(\sin\theta\cos 10° + \cos\theta\sin 10°) = 7(\cos\theta\cos 10° + \sin\theta\sin 10°)$$

$$3\sin\theta\cos 10° + 3\cos\theta\sin 10° = 7\cos\theta\cos 10° + 7\sin\theta\sin 10°$$

Divide through by $\cos\theta\cos 10°$:

$$3\tan\theta + 3\tan 10° = 7 + 7\tan\theta\tan 10°$$

$$3\tan\theta + 3p = 7 + 7p\tan\theta$$

$$\tan\theta(3-7p) = 7-3p$$

$$\tan\theta = \frac{7-3p}{3-7p}$$

Question 3 (Jun 2011, Q9)

Worked Solution

(i) Prove $\dfrac{\sin(\theta-\alpha)+3\sin\theta+\sin(\theta+\alpha)}{\cos(\theta-\alpha)+3\cos\theta+\cos(\theta+\alpha)} \equiv \tan\theta$**:**

Expand the numerator:

$$\sin(\theta-\alpha)+\sin(\theta+\alpha)=2\sin\theta\cos\alpha$$

So numerator $=2\sin\theta\cos\alpha+3\sin\theta=\sin\theta(2\cos\alpha+3)$.

Expand the denominator:

$$\cos(\theta-\alpha)+\cos(\theta+\alpha)=2\cos\theta\cos\alpha$$

So denominator $=2\cos\theta\cos\alpha+3\cos\theta=\cos\theta(2\cos\alpha+3)$.

Therefore:

$$\frac{\sin\theta(2\cos\alpha+3)}{\cos\theta(2\cos\alpha+3)}=\frac{\sin\theta}{\cos\theta}$$

$\equiv \tan\theta$ ✓

(ii) Exact value of $\dfrac{4\sin 149^\circ+12\sin 150^\circ+4\sin 151^\circ}{3\cos 149^\circ+9\cos 150^\circ+3\cos 151^\circ}$**:**

Factor out: numerator $=4(\sin 149^\circ+\sin 151^\circ)+12\sin 150^\circ$, denominator $=3(\cos 149^\circ+\cos 151^\circ)+9\cos 150^\circ$.

This matches the identity with $\theta=150^\circ$, $\alpha=1^\circ$, scaled by factor $\frac{4}{3}$:

Numerator: $4\times[\sin 149^\circ+3\sin 150^\circ+\sin 151^\circ]$. Denominator: $3\times[\cos 149^\circ+3\cos 150^\circ+\cos 151^\circ]$.

By part (i) with $\theta=150^\circ$, $\alpha=1^\circ$:

$$\frac{4[\sin(\theta-\alpha)+3\sin\theta+\sin(\theta+\alpha)]}{3[\cos(\theta-\alpha)+3\cos\theta+\cos(\theta+\alpha)]}=\frac{4}{3}\tan 150^\circ=\frac{4}{3}\times\left(-\frac{1}{\sqrt{3}}\right)$$

$$=-\frac{4}{3\sqrt{3}}=-\frac{4\sqrt{3}}{9}$$

(iii) Solve $\dfrac{\sin(6\theta-15^\circ)+3\sin 6\theta+\sin(6\theta+15^\circ)}{\cos(6\theta-15^\circ)+3\cos 6\theta+\cos(6\theta+15^\circ)}=k$ **for** $0^\circ<\theta<60^\circ$**:**

By part (i) with $\phi=6\theta$, $\alpha=15^\circ$, this equals $\tan 6\theta=k$.

So $6\theta=\tan^{-1}k$, giving $\theta=\frac{1}{6}\tan^{-1}k$.

A second solution in $0<6\theta<360^\circ$ uses $6\theta=\tan^{-1}k+180^\circ$:

$\theta=\frac{1}{6}\tan^{-1}k$ and $\theta=\frac{1}{6}\tan^{-1}k+30^\circ$

Question 4 (Jan 2013, Q9)

Worked Solution

(i) Prove $\cos^2(\theta+45°) - \frac{1}{2}(\cos 2\theta - \sin 2\theta) \equiv \sin^2\theta$**:**

Expand $\cos(\theta+45°) = \cos\theta\cos 45° - \sin\theta\sin 45° = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$:

$$\cos^2(\theta+45°) = \tfrac{1}{2}(\cos\theta-\sin\theta)^2 = \tfrac{1}{2}(\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta) = \tfrac{1}{2}(1-\sin 2\theta)$$

So the left-hand side:

$$\tfrac{1}{2}(1-\sin 2\theta) - \tfrac{1}{2}(\cos 2\theta - \sin 2\theta) = \tfrac{1}{2} - \tfrac{1}{2}\sin 2\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\sin 2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

$$= \tfrac{1}{2}(1-\cos 2\theta) = \sin^2\theta$$

$\equiv \sin^2\theta$ ✓

(ii) Solve $6\cos^2(\frac{1}{2}\theta+45°) - 3(\cos\theta - \sin\theta) = 2$ **for** $-90° < \theta < 90°$**:**

By part (i) with $\frac{\theta}{2}$ replacing θ: $\cos^2\left(\frac{\theta}{2}+45°\right) = \sin^2\frac{\theta}{2}$.

Also note $\cos\theta - \sin\theta = \cos 2\cdot\frac{\theta}{2} - \sin 2\cdot\frac{\theta}{2}$, but using the identity directly:

$6\sin^2\frac{\theta}{2} - 3(\cos\theta - \sin\theta) = 2$

Use $\cos\theta - \sin\theta$: from the identity, $\frac{1}{2}(\cos\theta - \sin\theta) = \frac{1}{2}\cos 2\theta - \frac{1}{2}\sin 2\theta \cdot$... Let's apply the identity from (i) with substitution $\frac{1}{2}\theta$:

$\cos^2(\frac{1}{2}\theta + 45°) = \sin^2(\frac{1}{2}\theta) = \frac{1}{2}(1-\cos\theta)$.

So: $6 \times \frac{1}{2}(1-\cos\theta) - 3(\cos\theta - \sin\theta) = 2$

$$3 - 3\cos\theta - 3\cos\theta + 3\sin\theta = 2$$

$$3\sin\theta - 6\cos\theta = -1$$

Hmm – let me re-examine. The question uses $6\cos^2(\frac{1}{2}\theta + 45°) - 3(\cos\theta - \sin\theta)$.

From (i) with $\theta \to \frac{1}{2}\theta$: left side $= 6\sin^2(\frac{1}{2}\theta)$.

Use $6\sin^2(\frac{1}{2}\theta) = 3(1-\cos\theta)$:

$$3(1-\cos\theta) - 3(\cos\theta - \sin\theta) = 2$$

$$3 - 3\cos\theta - 3\cos\theta + 3\sin\theta = 2$$

$$3\sin\theta - 6\cos\theta = -1$$

Use $\sin\frac{1}{2}\theta = c$ approach or directly: divide by $\cos\theta$... Actually use $\sin 2\phi = 2\sin\phi\cos\phi$. Rewrite:

$3\sin\theta - 6\cos\theta = -1 \implies \sin\theta - 2\cos\theta = -\frac{1}{3}$.

Let $\sin\frac{\theta}{2} = c$. Then $\sin\theta = 2c\sqrt{1-c^2}$, $\cos\theta = 1-2c^2$... this becomes complex. Instead use the direct result. We need $\sin\theta/2 = c$:

Actually the mark scheme says: use identity to get $\sin \frac{1}{2}\theta = c$ where c is a constant. The equation becomes $6\sin^2(\frac{1}{2}\theta) = 2$ after the subtraction cancels? Let me re-read the question.

The question is: $6\cos^2(\frac{1}{2}\theta + 45°) - 3(\cos\theta - \sin\theta) = 2$.

From (i) with $\theta \to \frac{1}{2}\theta$: $\cos^2(\frac{1}{2}\theta + 45°) - \frac{1}{2}(\cos\theta - \sin\theta) = \sin^2(\frac{1}{2}\theta)$.

So $\cos^2(\frac{1}{2}\theta + 45°) = \sin^2(\frac{1}{2}\theta) + \frac{1}{2}(\cos\theta - \sin\theta)$.

Substituting:

$$6\left[\sin^2(\tfrac{1}{2}\theta) + \tfrac{1}{2}(\cos\theta - \sin\theta)\right] - 3(\cos\theta - \sin\theta) = 2$$

$$6\sin^2(\tfrac{1}{2}\theta) + 3(\cos\theta - \sin\theta) - 3(\cos\theta - \sin\theta) = 2$$

$$6\sin^2(\tfrac{1}{2}\theta) = 2 \implies \sin^2(\tfrac{1}{2}\theta) = \tfrac{1}{3} \implies \sin(\tfrac{1}{2}\theta) = \pm\tfrac{1}{\sqrt{3}}$$

For $-90° < \theta < 90°$: $-45° < \frac{1}{2}\theta < 45°$.

$$\tfrac{1}{2}\theta = \pm\arcsin\left(\tfrac{1}{\sqrt{3}}\right) = \pm 35.26\ldots°$$

$$\theta = \pm 70.5° \quad \text{(to 1 d.p.)}$$

$\theta = 70.5°$ and $\theta = -70.5°$

(iii) Range of k for two solutions of $6\cos^2(\frac{1}{3}\theta + 45°) - 3(\cos\frac{2}{3}\theta - \sin\frac{2}{3}\theta) = k$ in $-90° < \theta < 90°$:

By the same working with $\theta \to \frac{1}{3}\theta$: $6\sin^2(\frac{1}{3}\theta) = k$, so $\sin^2(\frac{1}{3}\theta) = \frac{k}{6}$.

For $-90° < \theta < 90°$: $-30° < \frac{1}{3}\theta < 30°$, so $0 \le \sin^2(\frac{1}{3}\theta) < \frac{1}{4}$.

Two solutions (one positive and one negative angle) exist when $0 < \frac{k}{6} < \frac{1}{4}$, i.e. $0 < k < \frac{3}{2}$.

$$0 < k < \frac{3}{2}$$

Question 5 (Jun 2015, Q9)

Worked Solution

$f(\theta) = \sin(\theta + 30°) + \cos(\theta + 60°)$.

(i) Show $f(\theta) = \cos\theta$, then $f(4\theta) + 4f(2\theta) \equiv 8\cos^4\theta - 3$:

Expand using addition formulae:

$$f(\theta) = \sin\theta\cos 30° + \cos\theta\sin 30° + \cos\theta\cos 60° - \sin\theta\sin 60°$$

$$= \tfrac{\sqrt{3}}{2}\sin\theta + \tfrac{1}{2}\cos\theta + \tfrac{1}{2}\cos\theta - \tfrac{\sqrt{3}}{2}\sin\theta = \cos\theta$$

Now:

$$f(4\theta) + 4f(2\theta) = \cos 4\theta + 4\cos 2\theta$$

Use $\cos 4\theta = 2\cos^2 2\theta - 1$ and $\cos 2\theta = 2\cos^2\theta - 1$:

$$= 2\cos^2 2\theta - 1 + 4\cos 2\theta = 2(2\cos^2\theta - 1)^2 - 1 + 4(2\cos^2\theta - 1)$$

$$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 + 8\cos^2\theta - 4$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1 + 8\cos^2\theta - 4$$

$= 8\cos^4\theta - 3$ ✓

(ii)(a) Greatest and least values of $\dfrac{1}{f(4\theta) + 4f(2\theta) + 7}$:

From (i): $f(4\theta) + 4f(2\theta) = 8\cos^4\theta - 3$.

Range of $\cos^4\theta$: $0 \le \cos^4\theta \le 1$, so $8\cos^4\theta - 3 \in [-3, 5]$.

Denominator ranges over $[-3 + 7, 5 + 7] = [4, 12]$.

Greatest value: $\dfrac{1}{4}$ (when $\cos\theta = 0$)

Least value: $\dfrac{1}{12}$ (when $\cos\theta = \pm 1$)

(ii)(b) Solve $\sin(12\alpha + 30°) + \cos(12\alpha + 60°) + 4\sin(6\alpha + 30°) + 4\cos(6\alpha + 60°) = 1$ for $0° < \alpha < 60°$:

This is $f(12\alpha) + 4f(6\alpha) = 1$, i.e. $8\cos^4(3\alpha) - 3 = 1$:

$$8\cos^4(3\alpha) = 4 \implies \cos^4(3\alpha) = \tfrac{1}{2} \implies \cos^2(3\alpha) = \tfrac{1}{\sqrt{2}}$$

$$\cos(3\alpha) = \pm 2^{-1/4}$$

$3\alpha \in (0°, 180°)$: $\cos(3\alpha) = 2^{-1/4} = 0.8409\ldots \Rightarrow 3\alpha = 32.766\ldots° \Rightarrow \alpha = 10.9°$

$\cos(3\alpha) = -2^{-1/4} \Rightarrow 3\alpha = 180° - 32.77° = 147.23° \Rightarrow \alpha = 49.1°$

$\alpha = 10.9°$ and $\alpha = 49.1°$

Question 6 (Jan 2006, Q9)

Worked Solution

(i) Show $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$**:**

Write $\sin 3\theta = \sin(2\theta + \theta)$:

$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$$

$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta = 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ ✓

(ii) Greatest value of $9\sin(\frac{10}{3}\alpha) - 12\sin^3(\frac{10}{3}\alpha)$ **and smallest positive** α**:**

Factor: $9\sin(\frac{10}{3}\alpha) - 12\sin^3(\frac{10}{3}\alpha) = 3\left[3\sin(\frac{10}{3}\alpha) - 4\sin^3(\frac{10}{3}\alpha)\right] = 3\sin(3 \times \frac{10}{3}\alpha) = 3\sin(10\alpha)$.

The greatest value of $3\sin(10\alpha)$ is 3, occurring when $\sin(10\alpha) = 1$, i.e. $10\alpha = 90°$, so $\alpha = 9°$.

Greatest value: 3; smallest positive $\alpha = 9°$

(iii) Solve $3\sin 6\beta \operatorname{cosec} 2\beta = 4$ **for** $0° < \beta < 90°$**:**

Write $\operatorname{cosec} 2\beta = \frac{1}{\sin 2\beta}$:

$$\frac{3\sin 6\beta}{\sin 2\beta} = 4$$

Use $\sin 6\beta = \sin(3 \times 2\beta) = 3\sin 2\beta - 4\sin^3 2\beta$ (from part (i) with $\theta = 2\beta$):

$$\frac{3(3\sin 2\beta - 4\sin^3 2\beta)}{\sin 2\beta} = 4$$

$$3(3 - 4\sin^2 2\beta) = 4$$

$$9 - 12\sin^2 2\beta = 4 \implies \sin^2 2\beta = \frac{5}{12}$$

$$\sin 2\beta = \sqrt{\tfrac{5}{12}} \quad \text{(positive since } 0 < 2\beta < 180°\text{)}$$

$$2\beta = \arcsin\sqrt{\tfrac{5}{12}} = 40.2\ldots° \quad \text{or} \quad 2\beta = 180° - 40.2° = 139.8°$$

$$\beta = 20.1° \quad \text{or} \quad \beta = 69.9°$$

$\beta = 20.1°$ and $\beta = 69.9°$

Question 7 (Jun 2016, Q9)

Worked Solution

(i) Show $\sin 2\theta(\tan\theta + \cot\theta) \equiv 2$**:**

Write $\tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$.

Use $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\sin 2\theta(\tan\theta + \cot\theta) = 2\sin\theta\cos\theta \times \frac{1}{\sin\theta\cos\theta} = 2$$

$\equiv 2$ ✓

(ii)(a) Exact value of $\tan\frac{1}{12}\pi + \tan\frac{1}{8}\pi + \cot\frac{1}{12}\pi + \cot\frac{1}{8}\pi$**:**

Regroup: $(\tan\frac{\pi}{12} + \cot\frac{\pi}{12}) + (\tan\frac{\pi}{8} + \cot\frac{\pi}{8})$.

By part (i): $\tan\phi + \cot\phi = \dfrac{2}{\sin 2\phi}$.

$$= \frac{2}{\sin\frac{\pi}{6}} + \frac{2}{\sin\frac{\pi}{4}} = \frac{2}{\frac{1}{2}} + \frac{2}{\frac{\sqrt{2}}{2}} = 4 + \frac{4}{\sqrt{2}} = 4 + 2\sqrt{2}$$

$4 + 2\sqrt{2}$

(ii)(b) Solve $\sin 4\theta(\tan\theta + \cot\theta) = 1$ **for** $0 < \theta < \frac{1}{2}\pi$**:**

Write $\sin 4\theta(\tan\theta + \cot\theta) = \sin 4\theta \times \dfrac{2}{\sin 2\theta}$ (using part (i) result):

Actually using part (i): $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta} = \dfrac{2}{\sin 2\theta}$.

$\sin 4\theta \times \dfrac{2}{\sin 2\theta} = 1$. Use $\sin 4\theta = 2\sin 2\theta\cos 2\theta$:

$$\frac{2 \times 2\sin 2\theta\cos 2\theta}{\sin 2\theta} = 4\cos 2\theta = 1$$

$$\cos 2\theta = \tfrac{1}{4} \implies 2\theta = \arccos(\tfrac{1}{4}) = 75.52\ldots^\circ$$

$$\theta = 0.659\ldots \text{ rad}$$

(2θ in $(0, \pi)$, so one solution.)

$\theta = 0.659$ (or 0.66) radians

(ii)(c) Express $(1 - \cos 2\theta)^2(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta)^3$ **in terms of** $\sin\theta$**:**

$(1 - \cos 2\theta) = 2\sin^2\theta$, so $(1 - \cos 2\theta)^2 = 4\sin^4\theta$.

Using part (i) result with $\frac{1}{2}\theta$: $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = \dfrac{2}{\sin\theta}$.

$$4\sin^4\theta \times \left(\frac{2}{\sin\theta}\right)^3 = 4\sin^4\theta \times \frac{8}{\sin^3\theta} = \frac{32\sin\theta}{1}$$

$32\sin\theta$

Question 8 (Jun 2017, Q8)

Worked Solution

(i) Express $3\sin 2\theta \sec\theta + 4\sin 2\theta \operatorname{cosec}\theta$ **in the form** $R\sin(\theta+\alpha)$, $R > 0$, $0° < \alpha < 90°$**:**

Use $\sin 2\theta = 2\sin\theta\cos\theta$:

$$3\sin 2\theta \sec\theta + 4\sin 2\theta \operatorname{cosec}\theta = 3\sin 2\theta \cdot \frac{1}{\cos\theta} + 4\sin 2\theta \cdot \frac{1}{\sin\theta}$$

$$= \frac{3\times 2\sin\theta\cos\theta}{\cos\theta} + \frac{4\times 2\sin\theta\cos\theta}{\sin\theta} = 6\sin\theta + 8\cos\theta$$

Now write $6\sin\theta + 8\cos\theta = R\sin(\theta+\alpha)$:

$$R\sin(\theta+\alpha) = R\cos\alpha\sin\theta + R\sin\alpha\cos\theta$$

Matching: $R\cos\alpha = 6$, $R\sin\alpha = 8$.

$$R = \sqrt{6^2+8^2} = \sqrt{100} = 10$$

$$\tan\alpha = \tfrac{8}{6} = \tfrac{4}{3} \implies \alpha = 53.13\ldots°$$

$3\sin 2\theta \sec\theta + 4\sin 2\theta \operatorname{cosec}\theta \equiv 10\sin(\theta + 53.1°)$

(ii) Solve $3\sin(2\beta+20°)\sec(\beta+10°) + 4\sin(2\beta+20°)\operatorname{cosec}(\beta+10°) = 3$ **for** $0° < \beta < 360°$**:**

Let $\phi = \beta + 10°$. The expression becomes $3\sin 2\phi\sec\phi + 4\sin 2\phi \operatorname{cosec}\phi = 3$, which by part (i) equals $10\sin(\phi + 53.13°) = 3$:

$$\sin(\beta + 10° + 53.13°) = \frac{3}{10} = 0.3$$

$$\sin(\beta + 63.13°) = 0.3$$

$$\beta + 63.13° = \arcsin(0.3) = 17.46° \quad \text{or} \quad 180° - 17.46° = 162.54°$$

For $\beta \in (0°, 360°)$, $\beta + 63.13° \in (63.13°, 423.13°)$.

First solution: $\beta + 63.13° = 162.54° \Rightarrow \beta = 99.4°$

Second solution: $\beta + 63.13° = 360° + 17.46° = 377.46° \Rightarrow \beta = 314.3°$

$\beta = 99.4°$ and $\beta = 314°$ (both to nearest 0.1°)

End of Worked Solutions