## Question 1 (OCR 4722, Jan 2010, Q5)

### Worked Solution

The curves $y = x^2 + 1$ and $y = 11 - \frac{9}{x^2}$ intersect at $(1, 2)$ and $(3, 10)$.

The shaded area is:

$$\int_1^3 \left[\left(11 - \frac{9}{x^2}\right) - (x^2 + 1)\right] dx = \int_1^3 \left(10 - 9x^{-2} - x^2\right) dx$$

Integrating term by term:

$$= \left[10x + 9x^{-1} - \frac{x^3}{3}\right]_1^3$$

At $x = 3$: $30 + 3 - 9 = 24$

At $x = 1$: $10 + 9 - \frac{1}{3} = 19 - \frac{1}{3} = \frac{56}{3}$

Wait — let us recheck. The upper curve at any point between $x = 1$ and $x = 3$: at $x = 2$, $y_1 = 11 - 9/4 = 8.75$ and $y_2 = 5$, so $y = 11 - 9x^{-2}$ is on top.

$$\text{Area} = 24 - \frac{56}{3} = \frac{72 - 56}{3} = \frac{16}{3}$$

Alternatively, using the OR method from the mark scheme:

$$\left[11x + 9x^{-1}\right]_1^3 - \left[\frac{x^3}{3} + x\right]_1^3 = [(33 + 3) - (11 + 9)] - \left[(9 + 3) - \left(\frac{1}{3} + 1\right)\right] = 16 - \frac{32}{3} \cdot \frac{1}{\dots}$$

Let us compute directly:

$$\int_1^3 \left(10 - 9x^{-2} - x^2\right) dx = \left[10x + 9x^{-1} - \tfrac{1}{3}x^3\right]_1^3$$

At $x = 3$: $30 + 3 - 9 = 24$. At $x = 1$: $10 + 9 - \frac{1}{3} = \frac{57}{3} - \frac{1}{3} = \frac{56}{3}$.

$$\text{Area} = 24 - \frac{56}{3} = \frac{72 - 56}{3} = \frac{16}{3}$$

Hmm, the mark scheme gives $5\frac{1}{3}$. Let us recalculate: $24 - \frac{56}{3} = \frac{72}{3} - \frac{56}{3} = \frac{16}{3} = 5\frac{1}{3}$. ✓

Exact area $= \dfrac{16}{3} = 5\frac{1}{3}$

## Question 2 (OCR 4722, Jan 2012, Q7)

### Worked Solution

**Part (a): Find $\int (x^2+4)(x-6)\,dx$**

Expand: $(x^2+4)(x-6) = x^3 - 6x^2 + 4x - 24$.

Integrating:

$$\int (x^3 - 6x^2 + 4x - 24)\,dx = \frac{x^4}{4} - 2x^3 + 2x^2 - 24x + c$$

$$\frac{x^4}{4} - 2x^3 + 2x^2 - 24x + c$$

**Part (b): Find the shaded area enclosed by $y = 6x^{3/2}$, $y = \frac{8}{x^2} - 2$, and the $x$-axis**

The two curves intersect at $(1,6)$ (given). The curve $y = \frac{8}{x^2} - 2$ meets the $x$-axis when $\frac{8}{x^2} = 2 \Rightarrow x^2 = 4 \Rightarrow x = 2$ (using positive value). The curve $y = 6x^{3/2}$ passes through the origin.

The shaded region has two parts:

- From $x = 0$ to $x = 1$: area under $y = 6x^{3/2}$.
- From $x = 1$ to $x = 2$: area under $y = \frac{8}{x^2} - 2$.

**Left part** (0 to 1):

$$\int_0^1 6x^{3/2}\,dx = \left[\frac{6}{5/2}x^{5/2}\right]_0^1 = \left[\frac{12}{5}x^{5/2}\right]_0^1 = \frac{12}{5}$$

**Right part** (1 to 2):

$$\int_1^2 \left(\frac{8}{x^2} - 2\right) dx = \left[-8x^{-1} - 2x\right]_1^2 = (-4-4) - (-8-2) = -8 - (-10) = 2$$

Total area $= \frac{12}{5} + 2 = \frac{12}{5} + \frac{10}{5} = \frac{22}{5}$.

Total shaded area $= \frac{22}{5}$

## Question 3 (OCR 4723, Jun 2017, Q5)

### Worked Solution

The curve is $y = \sqrt{4x-3}$. The point $P(7,5)$ lies on the curve. Find the exact shaded area bounded by the curve, the normal at $P$, and the $x$-axis.

**Step 1: Find the gradient of the normal at $P$**

Differentiate: $y = (4x-3)^{1/2}$, so

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{-1/2} \times 4 = 2(4x-3)^{-1/2}$$

At $x = 7$: $\dfrac{dy}{dx} = 2(28-3)^{-1/2} = 2 \times \dfrac{1}{5} = \dfrac{2}{5}$.

Gradient of normal $= -\dfrac{5}{2}$.

**Step 2: Find where the normal meets the $x$-axis**

Equation of normal through $P(7,5)$:

$$y - 5 = -\frac{5}{2}(x-7) \implies y = -\frac{5}{2}x + \frac{35}{2} + 5 = -\frac{5}{2}x + \frac{45}{2}$$

Setting $y = 0$: $\dfrac{5}{2}x = \dfrac{45}{2} \Rightarrow x = 9$.

So the normal meets the $x$-axis at $(9,0)$.

**Step 3: Find the area under the curve from $x = \frac{3}{4}$ to $x = 7$**

(The curve meets the $x$-axis when $4x - 3 = 0 \Rightarrow x = \frac{3}{4}$.)

$$\int_{3/4}^{7} (4x-3)^{1/2}\,dx = \left[\frac{(4x-3)^{3/2}}{4 \times \frac{3}{2}}\right]_{3/4}^{7} = \left[\frac{(4x-3)^{3/2}}{6}\right]_{3/4}^{7}$$

At $x = 7$: $\dfrac{25^{3/2}}{6} = \dfrac{125}{6}$. At $x = \frac{3}{4}$: $\dfrac{0^{3/2}}{6} = 0$.

Area under curve $= \dfrac{125}{6}$.

**Step 4: Area of triangle formed by the normal**

The triangle has base from $x = 7$ to $x = 9$ along the $x$-axis (base $= 2$) and height $= 5$ (the $y$-value at $P$):

$$\text{Triangle area} = \frac{1}{2} \times 2 \times 5 = 5$$

**Step 5: Shaded area**

$$\text{Shaded area} = \frac{125}{6} + 5 = \frac{125}{6} + \frac{30}{6} = \frac{155}{6}$$

Exact shaded area $= \dfrac{155}{6}$

## Question 4 (OCR 4722, Jun 2017, Q6)

### Worked Solution

The curves $y = 11 - x - 2x^2$ and $y = \dfrac{8}{x^3}$ intersect at $(1, 8)$ and $(2, 1)$.

Shaded area $= \displaystyle\int_1^2 \left[(11 - x - 2x^2) - \frac{8}{x^3}\right] dx$

Integrate each term:

$$\int (11 - x - 2x^2)\,dx = 11x - \frac{x^2}{2} - \frac{2x^3}{3}$$

$$\int 8x^{-3}\,dx = \frac{8x^{-2}}{-2} = -4x^{-2}$$

So:

$$\text{Area} = \left[11x - \frac{x^2}{2} - \frac{2x^3}{3} - (-4x^{-2})\right]_1^2 = \left[11x - \frac{x^2}{2} - \frac{2x^3}{3} + 4x^{-2}\right]_1^2$$

At $x = 2$: $22 - 2 - \dfrac{16}{3} + 1 = 21 - \dfrac{16}{3} = \dfrac{63 - 16}{3} = \dfrac{47}{3}$

Wait — $4x^{-2}$ at $x = 2$ is $4/4 = 1$. Let me recompute:

At $x = 2$: $11(2) - \frac{4}{2} - \frac{2(8)}{3} + \frac{4}{4} = 22 - 2 - \frac{16}{3} + 1 = 21 - \frac{16}{3} = \frac{63-16}{3} = \frac{47}{3}$

Hmm, mark scheme separates the integrals. Let us follow the mark scheme method:

$$\int_1^2 (11 - x - 2x^2)\,dx = \left[11x - \tfrac{1}{2}x^2 - \tfrac{2}{3}x^3\right]_1^2 = \left(22 - 2 - \tfrac{16}{3}\right) - \left(11 - \tfrac{1}{2} - \tfrac{2}{3}\right)$$

$= \left(20 - \frac{16}{3}\right) - \left(\frac{66-3-4}{6}\right) = \frac{60-16}{3} - \frac{59}{6} = \frac{44}{3} - \frac{59}{6} = \frac{88-59}{6} = \frac{29}{6}$

$$\int_1^2 8x^{-3}\,dx = \left[-4x^{-2}\right]_1^2 = (-1) - (-4) = 3$$

$$\text{Area} = \frac{29}{6} - 3 = \frac{29 - 18}{6} = \frac{11}{6}$$

Exact shaded area $= \dfrac{11}{6}$

## Question 5 (OCR 4723, Jun 2016, Q5)

**Worked Solution**

Curves: $y = e^{2x}$ and $y = 8e^{-x}$. The shaded region is bounded by the two curves and the $y$-axis.

**Part (i): Show the curves intersect where $x = \ln 2$**

Set $e^{2x} = 8e^{-x}$:

$$e^{2x} \cdot e^{x} = 8 \implies e^{3x} = 8 \implies 3x = \ln 8 = 3\ln 2 \implies x = \ln 2$$

$x = \ln 2$ **(shown)**

**Part (ii): Find the area of the shaded region**

For $0 \le x \le \ln 2$: at $x = 0$, $y = 1$ (lower) and $y = 8$ (upper), so $y = 8e^{-x}$ is on top.

$$\text{Area} = \int_0^{\ln 2} \left(8e^{-x} - e^{2x}\right) dx = \left[-8e^{-x} - \frac{1}{2}e^{2x}\right]_0^{\ln 2}$$

At $x = \ln 2$: $-8e^{-\ln 2} - \frac{1}{2}e^{2\ln 2} = -8 \cdot \frac{1}{2} - \frac{1}{2} \cdot 4 = -4 - 2 = -6$

At $x = 0$: $-8 - \frac{1}{2}$

$$\text{Area} = (-6) - \left(-8 - \frac{1}{2}\right) = -6 + 8 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2}$$

Shaded area $= \dfrac{5}{2}$

## Question 6 (OCR 4723, Jun 2005, Q8)

### Worked Solution

Curves: $y = e^{x/5}$ and $y = \sqrt[3]{3x+8}$. They meet at point $P$ in the region $x > 0$.

**Part (i): Show that the $x$-coordinate of $P$ lies between 5.2 and 5.3**

At the intersection, $e^{x/5} = (3x+8)^{1/3}$. Define $f(x) = e^{x/5} - (3x+8)^{1/3}$.

At $x = 5.2$: $e^{1.04} \approx 2.829$, $(15.6+8)^{1/3} = 23.6^{1/3} \approx 2.868$. So $f(5.2) \approx 2.829 - 2.868 = -0.039 < 0$.

At $x = 5.3$: $e^{1.06} \approx 2.887$, $(15.9+8)^{1/3} = 23.9^{1/3} \approx 2.882$. So $f(5.3) \approx 2.887 - 2.882 = 0.005 > 0$.

Since $f$ is continuous and changes sign in $[5.2, 5.3]$, there is a root in this interval. **(shown)**

**Part (ii): Show the $x$-coordinate satisfies $x = \frac{5}{3}\ln(3x+8)$**

At the intersection: $e^{x/5} = (3x+8)^{1/3}$.

Taking the natural log: $\dfrac{x}{5} = \dfrac{1}{3}\ln(3x+8) \implies x = \dfrac{5}{3}\ln(3x+8)$. **(shown)**

**Part (iii): Use the iterative formula to find $x$ to 2 d.p.**

The iterative formula is $x_{n+1} = \dfrac{5}{3}\ln(3x_n + 8)$.

Starting from $x_0 = 5.2$:

$$
\begin{aligned}
x_1 &= \tfrac{5}{3}\ln(3(5.2)+8) = \tfrac{5}{3}\ln(23.6) \approx \tfrac{5}{3}(3.1612) \approx 5.2687 \\
x_2 &= \tfrac{5}{3}\ln(3(5.2687)+8) \approx \tfrac{5}{3}\ln(23.806) \approx 5.2832 \\
x_3 &= \tfrac{5}{3}\ln(23.850) \approx 5.2863 \\
x_4 &= \tfrac{5}{3}\ln(23.859) \approx 5.2869 \\
x_5 &\approx 5.2870
\end{aligned}
$$

The sequence converges to $x \approx 5.29$ (to 2 d.p.).

$x$-coordinate of $P \approx$ **5.29**

**Part (iv): Approximate area of region $R$**

Region $R$ is bounded by the two curves and the $y$-axis, from $x = 0$ to $x \approx 5.29$.

In this region, $y = (3x+8)^{1/3}$ is above $y = e^{x/5}$ (as $f(5.2) < 0$ shows).

$$\text{Area} = \int_0^{5.29} \left[(3x+8)^{1/3} - e^{x/5}\right] dx$$

Integrate $(3x+8)^{1/3}$: Let $u = 3x+8$, $du = 3\,dx$:

$$\int (3x+8)^{1/3} dx = \frac{1}{3} \cdot \frac{(3x+8)^{4/3}}{4/3} = \frac{(3x+8)^{4/3}}{4}$$

Integrate $e^{x/5}$: $\int e^{x/5}dx = 5e^{x/5}$.

$$\text{Area} = \left[\frac{(3x+8)^{4/3}}{4} - 5e^{x/5}\right]_0^{5.29}$$

At $x = 5.29$: $(3(5.29)+8)^{1/3} \approx 23.87^{1/3} \approx 2.877$, so $(23.87)^{4/3} \approx 23.87 \times 2.877 \approx 68.67$; $\frac{68.67}{4} \approx 17.17$. Also $5e^{5.29/5} \approx 5e^{1.058} \approx 5(2.880) \approx 14.40$.

At $x = 0$: $\frac{8^{4/3}}{4} - 5 = \frac{16}{4} - 5 = 4 - 5 = -1$.

$$\text{Area} \approx (17.17 - 14.40) - (-1) = 2.77 + 1 = 3.77$$

Approximate area of region $R \approx \mathbf{3.8}$ (2 s.f.)

End of Worked Solutions