Question 1 (Jun 2010, Q12)

Worked Solution

Part (i)(A): Number of buds with 10 stages of nodes

Each node has 2 stems growing from it, and at the end of each final stem there is a bud. At stage n, the number of buds equals the number of final stems, which equals 2^n.

With $n = 10$:

$$\text{Buds} = 2^{10} = 1024$$

Number of buds = **1024**

Part (i)(B): Number of stems with 10 stages of nodes

At each stage k, there are 2^k new stems. The total number of stems is the sum over all stages:

$$\text{Total stems} = 1 + 2 + 4 + \cdots + 2^{10} = 2^{11} - 1 = 2047$$

(using the geometric series formula $S = \frac{a(r^n-1)}{r-1}$ with $a = 1$, $r = 2$, $n = 11$ terms).

Number of stems = **2047**

Part (ii)(A): Show that the number of leaves with n stages is $7(2^n - 1)$

At stage k, there are 2^{k-1} nodes (since stage 1 has 1 node, stage 2 has 2, . . . , stage k has 2^{k-1}). Each node has 7 leaves, so the total number of leaves is:

$$7 \times (1 + 2 + 4 + \cdots + 2^{n-1}) = 7 \times \frac{2^n - 1}{2 - 1} = 7(2^n - 1)$$

Number of leaves = $7(2^n - 1)$ **(shown)**

Part (ii)(B): Find least n such that leaves $> 200\,000$

We need $7(2^n - 1) > 200\,000$, i.e.:

$$2^n - 1 > \frac{200\,000}{7} \implies 2^n > \frac{200\,000}{7} + 1 = \frac{200\,007}{7}$$

Taking $\log_{10}$ of both sides:

$$n \log_{10} 2 > \log_{10}\left(\frac{200\,007}{7}\right) = \log_{10} 200\,007 - \log_{10} 7$$

$$n > \frac{\log_{10} 200\,007 - \log_{10} 7}{\log_{10} 2}$$

Evaluating: $\log_{10} 200\,007 \approx 5.3011$, $\log_{10} 7 \approx 0.8451$, $\log_{10} 2 \approx 0.3010$:

$$n > \frac{5.3011 - 0.8451}{0.3010} = \frac{4.4560}{0.3010} \approx 14.80$$

Since n must be a positive integer:

Least possible value of $n = \mathbf{15}$

Question 2 (Jun 2005, Q11)

Worked Solution

Part (i): Number of flowerheads in year 5

In year n there are 3^{n-1} flowerheads (year 1: $3^0 = 1$, year 2: $3^1 = 3$, year 3: $3^2 = 9$, ...).

Year 5: $3^4 = 81$.

Flowerheads in year 5 = **81**

Part (ii): Number of flowerheads in year n

Flowerheads in year $n = \mathbf{3^{n-1}}$

Part (iii): Show total stems in year n is $\dfrac{3^n - 1}{2}$

The total stems form a geometric series: $1 + 3 + 9 + \cdots + 3^{n-1}$, with first term $a = 1$ and common ratio $r = 3$.

Using the GP sum formula:

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{1 \cdot (3^n - 1)}{3 - 1} = \frac{3^n - 1}{2}$$

Total stems in year $n = \dfrac{3^n - 1}{2}$ **(shown)**

Part (iv)(A): Age of Kitty's oleander (total stems = 364)

Set $\dfrac{3^n - 1}{2} = 364$:

$$3^n - 1 = 728 \implies 3^n = 729 = 3^6 \implies n = 6$$

Age = **6** years

Part (iv)(B): Number of flowerheads

Flowerheads $= 3^{n-1} = 3^5 = 243$.

Number of flowerheads = **243**

Part (v): Abdul's oleander has over 900 flowerheads; show $y > \dfrac{\log_{10} 900}{\log_{10} 3} + 1$

Flowerheads in year y is 3^{y-1}. We need $3^{y-1} > 900$:

$$(y - 1)\log_{10} 3 > \log_{10} 900 \implies y - 1 > \frac{\log_{10} 900}{\log_{10} 3} \implies y > \frac{\log_{10} 900}{\log_{10} 3} + 1$$

Evaluating: $\dfrac{\log_{10} 900}{\log_{10} 3} = \dfrac{2.9542}{0.4771} \approx 6.192$, so $y > 7.192$.

Since y must be an integer:

Smallest integer value of $y = \mathbf{8}$

Question 3 (Jun 2007, Q11)

Worked Solution

Part (a)(i): Counters in sixth pile

This is an arithmetic sequence with first term $a = 3$ and common difference $d = 2$.

The nth term is $a + (n-1)d$, so the 6th term:

$$3 + 5 \times 2 = 13$$

Counters in 6th pile = **13**

Part (a)(ii): Total counters in 10 piles

Using the arithmetic series sum formula $S_n = \frac{n}{2}(2a + (n-1)d)$:

$$S_{10} = \frac{10}{2}(2 \times 3 + 9 \times 2) = 5(6 + 18) = 5 \times 24 = 120$$

Total counters = **120**

Part (b)(i): Calculate P_4

$$P_4 = \frac{1}{6} \times \left(\frac{5}{6}\right)^3 = \frac{1}{6} \times \frac{125}{216} = \frac{125}{1296}$$

$$P_4 = \frac{125}{1296}$$

Part (b)(ii): First term, common ratio, and show $P_1 + P_2 + P_3 + \cdots = 1$

The values $P_1, P_2, P_3, \ldots$ form a GP with:

- First term $a = P_1 = \frac{1}{6}$
- Common ratio $r = \frac{5}{6}$

Since $|r| < 1$, the infinite sum converges:

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{6}}{1 - \frac{5}{6}} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1$$

$P_1 + P_2 + P_3 + \cdots = \mathbf{1}$ **(shown)**

Part (b)(iii): Find least n such that $P_n < 0.001$

We need $\frac{1}{6}\left(\frac{5}{6}\right)^{n-1} < 0.001$, i.e.:

$$\left(\frac{5}{6}\right)^{n-1} < 0.006$$

Taking $\log_{10}$: since $\log_{10}\left(\frac{5}{6}\right) < 0$, the inequality reverses when dividing:

$$(n-1)\log_{10}\left(\frac{5}{6}\right) < \log_{10} 0.006 \implies n-1 > \frac{\log_{10} 0.006}{\log_{10}\left(\frac{5}{6}\right)} \implies n > \frac{\log_{10} 0.006}{\log_{10}\left(\frac{5}{6}\right)} + 1$$

Evaluating: $\log_{10} 0.006 \approx -2.2218$, $\log_{10}(5/6) \approx -0.07918$:

$$n > \frac{-2.2218}{-0.07918} + 1 = 28.06 + 1 = 29.06$$

Least value of $n =$ **30**

Question 4 (Jun 2011, Q12)

Worked Solution

Part (i)(A): Jim's 12th payment

Jim's payments form an AP with $a = 500$, $d = -10$.

12th payment $= 500 + 11 \times (-10) = 500 - 110 = 390$.

Jim's 12th payment $=$ **£390**

Part (i)(B): Show Jim pays £9240 in 24 payments

Using $S_n = \frac{n}{2}(2a + (n-1)d)$ with $n = 24$, $a = 500$, $d = -10$:

$$S_{24} = \frac{24}{2}(2 \times 500 + 23 \times (-10)) = 12(1000 - 230) = 12 \times 770 = 9240$$

Jim pays £**9240** in total **(shown)**

Part (ii)(A): Mary's 12th payment

Mary's payments form a GP with $a = 460$, $r = 0.98$.

12th payment $= 460 \times 0.98^{11} \approx 460 \times 0.8007 \approx$ £368.33.

Mary's 12th payment $\approx$ **£368.33**

Part (ii)(B): Show Jim's 20th payment < Mary's 20th, but Jim's 19th ≥ Mary's 19th

Jim's nth payment: $J_n = 500 - 10(n-1)$. Mary's nth payment: $M_n = 460 \times 0.98^{n-1}$.

For $n = 20$:

$$J_{20} = 500 - 190 = 310, \quad M_{20} = 460 \times 0.98^{19} \approx 460 \times 0.6812 \approx 313.4$$

So $J_{20} = 310 < M_{20} \approx 313.4$. ✓

For $n = 19$:

$$J_{19} = 500 - 180 = 320, \quad M_{19} = 460 \times 0.98^{18} \approx 460 \times 0.6951 \approx 319.8$$

So $J_{19} = 320 \geq M_{19} \approx 319.8$. ✓

$J_{20} \approx 310 < M_{20} \approx 313.4$ and $J_{19} = 320 \geq M_{19} \approx 319.8$ **(shown)**

Part (ii)(C): Mary's total over 24 payments

Using GP sum $S_n = \frac{a(1 - r^n)}{1 - r}$ with $a = 460$, $r = 0.98$, $n = 24$:

$$S_{24} = \frac{460(1 - 0.98^{24})}{1 - 0.98} = \frac{460(1 - 0.98^{24})}{0.02}$$

$0.98^{24} \approx 0.6080$, so:

$$S_{24} \approx \frac{460 \times 0.3920}{0.02} = \frac{180.32}{0.02} \approx 8837$$

Mary's total $\approx$ £**8837**

Part (ii)(D): Find Mary's first payment a so she pays same total (£9240) over 24 months

Set $\dfrac{a(1-0.98^{24})}{1-0.98} = 9240$:

$$a = \frac{9240 \times 0.02}{1-0.98^{24}} = \frac{184.8}{0.3920} \approx 471.4$$

Mary's first payment $\approx$ £**481** (to nearest penny: $\approx$ £480.97)

Question 5 (Jun 2009, Q11)

Worked Solution

Part (i)(A): Show Haroon receives £210 for 6 correct answers

The prize amounts form an AP: $£10, £20, £30, \ldots$ with $a = 10$, $d = 10$, $n = 6$:

$$S_6 = \frac{6}{2}(10 + 60) = 3 \times 70 = 210$$

Haroon receives $£\mathbf{210}$ **(shown)**

Part (i)(B): Formula for total after n correct answers; find n for £10350

Total $= \frac{n}{2}(10 + 10n) = \frac{n}{2} \times 10(1 + n) = 5n(n + 1)$.

Set $5n(n + 1) = 10350$:

$$n(n + 1) = 2070 \implies n^2 + n - 2070 = 0 \implies (n - 45)(n + 46) = 0 \implies n = 45$$

Total $= 5n(n + 1)$; $n = \mathbf{45}$

Part (ii)(A): How many questions did Gary get right? (£75 received)

In the Double Your Money game, prizes are $£5, £10, £20, \ldots$ (GP with $a = 5$, $r = 2$).

Total after n correct: $S_n = 5(2^n - 1)$.

Set $5(2^n - 1) = 75 \Rightarrow 2^n - 1 = 15 \Rightarrow 2^n = 16 \Rightarrow n = 4$.

Gary got **4** questions right.

Part (ii)(B): Bethan's total for 9 correct answers

$$S_9 = 5(2^9 - 1) = 5 \times 511 = £2555$$

Bethan received $£\mathbf{2555}$

Part (ii)(C): Formula and find n for £2621435

Total $= 5(2^n - 1)$.

Set $5(2^n - 1) = 2\,621\,435$:

$$2^n - 1 = 524\,287 \implies 2^n = 524\,288$$

Taking $\log_{10}$: $n = \log_2 524\,288 = \dfrac{\log_{10} 524\,288}{\log_{10} 2} \approx \dfrac{5.7196}{0.3010} \approx 19$

Check: $2^{19} = 524\,288$. ✓

Total $= 5(2^n - 1)$; $n = \mathbf{19}$

Question 6 (Jun 2015, Q11)

Worked Solution

Part (i): Descendants in generation 8

Each generation multiplies by 3. Generation n has 3^n descendants.

Generation 8: $3^8 = 6561$.

Descendants in generation 8 = **6561**

Part (ii): Total descendants in first 15 generations

Sum of GP with $a = 3$, $r = 3$, $n = 15$:

$$S_{15} = \frac{3(3^{15} - 1)}{3 - 1} = \frac{3^{16} - 3}{2} = \frac{43\,046\,721 - 3}{2} = 21\,523\,359$$

Total descendants = **21 523 359**

Part (iii): Show $n > \dfrac{\log_{10} 2000003}{\log_{10} 3} - 1$; find least n

Total after n generations $= \dfrac{3(3^n - 1)}{2} = \dfrac{3^{n+1} - 3}{2}$.

Set $> 1\,000\,000$:

$$3^{n+1} - 3 > 2\,000\,000 \implies 3^{n+1} > 2\,000\,003$$

Taking $\log_{10}$:

$$(n+1)\log_{10} 3 > \log_{10} 2\,000\,003 \implies n+1 > \frac{\log_{10} 2\,000\,003}{\log_{10} 3} \implies n > \frac{\log_{10} 2\,000\,003}{\log_{10} 3} - 1$$

Evaluating: $\dfrac{\log_{10} 2\,000\,003}{\log_{10} 3} \approx \dfrac{6.3010}{0.4771} \approx 13.21$, so $n > 12.21$.

Least value of n = **13**

Part (iv): How many fewer descendants in 15 generations with 2 daughters instead of 3?

With 2 daughters: each generation n has 2^n descendants.

Total over 15 generations $= \dfrac{2(2^{15} - 1)}{2 - 1} = 2(32\,767) = 65\,534$.

Difference $= 21\,523\,359 - 65\,534 = 21\,457\,825$.

Jill would have **21 457 825** fewer descendants.

End of Worked Solutions